HIGHER ALGEBRA

WENTWORTH

A HIGHER ALGEBRA

BY

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PREFACE.

THIS work is intended to give in one book a thorough preparatory course for Colleges and Scientific Schools, and in addition a sufficiently full treatment of the subjects usually read by students in general in such institutions. In short, it provides a course parallel to the course covered by the author's School and College Algebras together. The elementary part is as full as the School Algebra; the advanced part, however, is briefer than the College Algebra. The book is substantially equivalent to the author's Complete Algebra, but is greatly superior to that work in the arrangement of topics and in the methods of presenting them.

Preparatory Schools and Academies, if their pupils have had a thorough drill in Arithmetic before they begin the study of Algebra, will find this book specially suited to their needs. The brighter boys of the class can read the advanced chapters while the duller boys review the elementary chapters.

Colleges and Scientific Schools, if their pupils, owing to lack of previous drill, have to review carefully the preparatory work of the schools before entering upon the college work proper, will find in this a convenient and sufficiently full book for their requirements.

Answers to the problems are bound separately, in paper covers, and will be furnished free for pupils when *teachers* apply to the publishers for them.

Any corrections or suggestions relating to the work will be thankfully received.

G. A. WENTWORTH.

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HIGHER ALGEBRA.

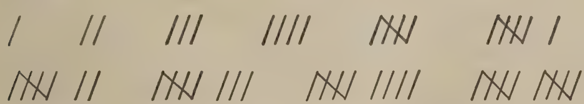
CHAPTER I.

DEFINITIONS.

1. Units. In counting separate objects the standards by which we count are called **units**; and in measuring continuous magnitudes the standards by which we measure are called **units.**

Thus, in counting the boys in a school, the unit is a boy; in selling eggs by the dozen, the unit is a dozen eggs; in selling bricks by the thousand, the unit is a thousand bricks; in measuring short distances, the unit is an inch, a foot, or a yard; in measuring long distances, the unit is a rod or a mile.

2. Numbers. Repetitions of the unit are expressed by numbers. If a man, in sawing logs into boards, wishes to keep a count of the logs, he makes a straight mark for every log sawed, and his record at different times will be as follows:



These representative groups are named one, two, three, four, five, six, seven, eight, nine, ten, etc., and are known collectively under the general name of **numbers.** It is obvious that these representative groups will have the same meaning, whatever the nature of the unit counted.

3. Quantities. The word "quantity" (from the Latin *quantus*, how much) implies both a unit and a number. Thus, if we inquire how much wheat a bin will hold, we mean how many bushels of wheat it will hold.

4. Number-Symbols in Arithmetic. Instead of groups of straight marks, we use in Arithmetic the arbitrary symbols 1, 2, 3, 4, 5, 6, 7, 8, 9, called **figures,** for the numbers one, two, three, four, five, six, seven, eight, nine.

The next number, ten, is indicated by writing the figure 1 in a different position, so that it shall signify not *one*, but *ten*. This change of position is effected by introducing a new symbol, 0, called **nought** or **zero,** and signifying *none*. Thus, in the symbol 10, the figure 1, occupying the second place from the right, signifies a collection of *ten* things, and the zero signifies that there are no single things over. The symbol 11 denotes a collection of ten things and one thing besides. All succeeding numbers up to the number consisting of 10 tens are expressed by writing the figure for the number of tens they contain in the second place from the right, and the figure for the number of units besides in the first place. The number consisting of 10 tens is called a **hundred,** and the *hundreds* of a number are written in the *third place* from the right. The number consisting of 10 hundreds is called a **thousand,** and the *thousands* are written in the *fourth place* from the right; and so on.

5. Number-Symbols in Algebra. Algebra, like Arithmetic, *treats of numbers*, and employs *the letters of the alphabet* in addition to the figures of Arithmetic to represent numbers. The letters of the alphabet are used as *general* symbols of numbers to which *any particular values* may be assigned. In any problem, however, a letter must be supposed to have the same particular value throughout the investigation or discussion of the problem.

These general symbols are of great advantage in investigating and stating general laws; in exhibiting the actual method in which a number is made up; and in representing *unknown numbers* which are to be discovered from their relations to known numbers.

6. Principal Signs of Operations. The signs of the fundamental operations are the same in Algebra as in Arithmetic, and are

The sign $+$ (read *plus*), the sign of addition.

The sign $-$ (read *minus*), the sign of subtraction.

The sign $\times$ (read *times* or *into*), the sign of multiplication.

The sign $\div$ (read *divided by*), the sign of division.

The multiplier is sometimes written after the sign, and then the sign is read *multiplied by*.

The operation of division is often indicated by writing the dividend over the divisor with a line between them. Thus $\frac{8}{4}$ means the same as $8 \div 4$.

7. Signs of Relation. The signs of relation are $=$, $>$, $<$, which stand for the words, "is equal to," "is greater than," and "is less than," respectively.

8. Signs of Aggregation. The signs of aggregation are the bar, $|$; the vinculum, $\overline{\quad}$; the parenthesis, ( ); the bracket, [ ]; and the brace, { }. Thus, each of the expressions, $\left.\begin{matrix} a \\ +b \end{matrix}\right|$, $\overline{a+b}$, $(a+b)$, $[a+b]$, $\{a+b\}$, signifies that $a+b$ is to be treated as a single number.

9. Signs of Continuation. The signs of continuation are dots, ..., or dashes, – – — – –, and are read, "and so on."

10. Sign of Deduction. The sign of deduction is $\therefore$, and is read, "hence," or "therefore."

11. The Natural Series of Numbers. Beginning with the number *one*, each succeeding number is obtained by putting one more with the preceding number.

Thus, if from a given point marked 0, we draw a straight line to the right, and beginning from this point lay off units of length, the successive repetitions of the unit will be denoted by the natural series of numbers, 1, 2, 3, 4, etc.

12. Positive and Negative Numbers. There are quantities which stand to each other in such opposite relations that, when we combine them, they cancel each other entirely or in part. Thus, six dollars *gain* and six dollars *loss* just cancel each other; but ten dollars *gain* and six dollars *loss* cancel each other only in part. For the six dollars *loss* will cancel six dollars of the *gain* and leave four dollars gain.

An opposition of this kind exists in *assets* and *debts*, in motion *forwards* and motion *backwards*, in motion *to the right* and motion *to the left*, in the degrees *above* and the degrees *below* zero on a thermometer.

From this relation of quantities a question often arises which is not considered in Arithmetic; namely, the subtracting of a greater number from a smaller. This cannot be done in Arithmetic, for the real nature of subtraction consists in *counting backwards*, along the natural series of numbers. If we wish to subtract four from six, we start at six in the natural series, count four units backwards, and arrive at two, the difference sought. If we subtract six from six, we start at six in the natural series, count six units backwards, and arrive at zero. If we try to subtract

nine from six, we cannot do it, because, when we have counted backwards as far as zero, *the natural series of numbers comes to an end.*

13. In order to subtract a greater number from a smaller, it is necessary to *assume* a new series of numbers, beginning at zero and extending to the left of zero. The series to the left of zero must ascend from zero by the repetitions of the unit, precisely like the natural series to the right of zero; and the *opposition* between the right-hand series and the left-hand series must be clearly marked. This opposition is indicated by calling every number in the right-hand series a *positive* number, and prefixing to it, when written, the sign $+$; and by calling every number in the left-hand series a *negative* number, and prefixing to it the sign $-$. The two series of numbers will be written thus:

$$\ldots\ldots -4,\ -3,\ -2,\ -1,\ \mathbf{0},\ +1,\ +2,\ +3,\ +4,\ \ldots\ldots$$

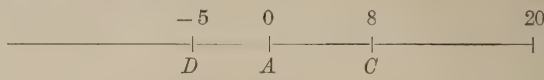
If, now, we wish to subtract 9 from 6, we begin at 6 in the positive series, count nine units in the *negative direction* (to the left), and arrive at -3 in the negative series. That is, $6-9=-3$.

The result obtained by subtracting a greater number from a less, when both are positive, is *always a negative number.*

If a and b represent any two numbers of the positive series, the expression $a-b$ will denote a positive number when a is greater than b; will be equal to zero when a is equal to b; will denote a negative number when a is less than b.

If we wish to add 9 to -6, we begin at -6, in the negative series, count nine units in the *positive direction* (to the right), and arrive at $+3$, in the positive series.

We may illustrate the use of positive and negative numbers as follows:



Suppose a person starting at A walks 20 feet to the right of A, and then returns 12 feet, where will he be? Answer: at C, a point 8 feet to the right of A. That is, 20 feet − 12 feet equals 8 feet.

Again, suppose he walks from A to the right 20 feet, and then returns 25 feet, where will he now be? Answer: at D, a point 5 feet to the left of A. That is, if we consider distances measured in feet to the left of A as expressed by a negative series of numbers, beginning at A, 20 feet − 25 feet equals −5 feet. Hence, the phrase, 5 feet to the left of A, is now expressed by the negative quantity, −5 feet.

REMARK. In Arithmetic, if the things counted are *whole units*, the numbers which count them are called **whole numbers, integral numbers**, or **integers**, where the adjective is transferred from the things counted to the numbers which count them. But if the things counted are only *parts of units*, the numbers which count them are called **fractional numbers**, or simply **fractions**, where again the adjective is transferred from the things counted to the numbers which count them.

In Algebra, if the units counted are *negative*, the numbers which count them are called **negative numbers**, where the adjective which defines the nature of the units counted is transferred to the numbers that count them.

14. Numbers with the sign + or − are called **algebraic numbers.** They are unknown in Arithmetic, but play a very important part in Algebra. Numbers not affected by the signs + or − are called **absolute numbers.**

Every algebraic number, as +4 or −4, consists of a sign + or − and the absolute value of the number; in this case 4. The sign shows whether the number belongs to

the positive or negative series of numbers; the absolute value shows the place the number has in the positive or negative series.

When no sign stands before a number, the sign $+$ is always understood; thus, 4 means the same as $+4$, a means the same as $+a$. But *the sign $-$ is never omitted.*

Two numbers which have, one the sign $+$ and the other the sign $-$, are said to have **unlike signs.**

Two numbers which have the same absolute values, but unlike signs, always cancel each other when combined; thus, $+4-4=0$, $+a-a=0$.

15. Meaning of the Signs. The use of the signs $+$ and $-$, to indicate addition and subtraction, must be carefully distinguished from their use to indicate in which series, the positive or the negative, a given number belongs. In the first sense, they are signs of *operations*, and are common to both Arithmetic and Algebra. In the second sense, they are signs of *opposition*, and are employed in Algebra alone.

16. Factors. When a number consists of the product of two or more numbers, each of these numbers is called a **factor** of the product.

The sign $\times$ is generally omitted between a figure and a letter, or between letters; thus, instead of $63 \times a \times b$, we write $63ab$; instead of $a \times b \times c$, we write abc.

The expression abc must not be confounded with $a+b+c$.

If $\quad a=2, \quad b=3, \quad c=4,$

then $\quad abc = 2 \times 3 \times 4 = 24;$

and $\quad a+b+c=2+3+4=9.$

Factors expressed by letters are called **literal** factors; factors expressed by figures are called **numerical** factors.

17. Coefficients. A known factor of a product which is prefixed to another factor, to show the number of times that factor is taken, is called a **coefficient**. Thus, in $7c$, 7 is the coefficient of c; in $7ax$, 7 is the coefficient of ax, or, if a is known, $7a$ is the coefficient of x.

By coefficient, we generally mean the *numerical* coefficient with its *sign*. If no numerical coefficient is written, 1 is understood. Thus, ax means the same as $1ax$.

18. Powers. A product consisting of two or more equal factors is called a **power** of that factor.

19. Indices or **Exponents.** An index or exponent is a number-symbol written at the right of, and a little above, a number.

If the index is a *positive integral number*, it shows the number of times the given number is taken as a factor.

Thus, a^1, or simply a, denotes that a is taken *once* as a factor; a^2 denotes that a is taken *twice* as a factor; a^3 denotes that a is taken *three times* as a factor; and a^n denotes that a is taken n *times* as a factor. These are read: the first power of a; the second power of a; the third power of a; the nth power of a.

a^3 is written instead of aaa.

a^n is written instead of aaa, etc., repeated n times.

The meaning of coefficient and exponent must be carefully distinguished. Thus,

$$4a = a + a + a + a;$$

$$a^4 = a \times a \times a \times a.$$

If $a = 3$, $$4a = 3 + 3 + 3 + 3 = 12.$$

$$a^4 = 3 \times 3 \times 3 \times 3 = 81.$$

The second power of a number is generally called the *square* of that number; thus, a^2 is called the *square* of a, because if a denotes the number of units of length in the side of a square, a^2 denotes the number of units of surface in the square. The third power of a number is generally called the *cube* of that number; thus, a^3 is called the *cube* of a, because if a denotes the number of units of length in the edge of a cube, a^3 denotes the number of units of volume in the cube.

20. Roots. The **root** of a number is one of the equal factors of that number; the *square root* of a number is one of the *two* equal factors of that number; the *cube root* of a number is one of the *three* equal factors of that number; and so on. The sign $\sqrt{\ }$, called the **radical sign,** indicates that a root is to be found. Thus, $\sqrt[2]{4}$, or $\sqrt{4}$, means that the square root of 4 is to be taken; $\sqrt[3]{8}$ means that the cube root of 8 is to be taken; and so on. The symbol written above the radical sign is called the **index of the root.**

21. Algebraic Expressions. An algebraic expression is a number written with algebraic symbols; an algebraic expression consists of one symbol, or of several symbols connected by signs of operation.

A **term** is an algebraic expression the parts of which are not separated by the sign of addition or subtraction. Thus, $3ab$, $5xy$, $3ab \times 5xy$, $3ab \div 5xy$ are terms.

A **monomial** or **simple expression** is an expression with but one term.

A **polynomial** or **compound expression** is an expression of two or more terms. A **binomial** is a polynomial of two terms; a **trinomial,** a polynomial of three terms.

Like terms or **similar terms** are terms which have the same letters, and the corresponding letters affected by the same exponents. Thus, $7a^2cx^3$ and $-5a^2cx^3$ are like terms.

22. The **degree** of a term is the sum of the exponents of its literal factors. Thus, $3xy$ is of the *second* degree.

A polynomial is said to be **homogeneous** when all its terms are of the same degree. Thus, $7x^3-5x^2y+xyz$ is homogeneous of the third degree.

A polynomial is said to be **arranged** according to the powers of some letter when the exponents of that letter either descend or ascend in order of magnitude.

23. The **value** of an algebraic expression is the number which the expression represents.

Exercise 1.

If $a = 1$, $b = 2$, $c = 3$, $d = 4$, $e = 5$, $f = 0$, find the values of the following expressions:

1. $9a + 2b + 3c - 2f$.

2. $4e - 3a - 3b + 5c$.

3. $8abc - bcd + 9cde - def$.

4. $\frac{4ac}{b} + \frac{8bc}{d} - \frac{5cd}{e}$.

5. $7e + bcd - \frac{3bde}{2ac}$.

6. $abc^2 + bcd^2 - dea^2 + f^3$.

7. $e^4 + 6e^2b^2 + b^4 - 4e^3b - 4eb^3$.

8. $\frac{8a^2 + 3b^2}{a^2b^2} + \frac{4c^2 + 6b^2}{c^2 - b^2} - \frac{c^2 + d^2}{e^2}$.

9. $\frac{d^c}{b^e}$.

10. $\frac{e^c + b^a}{c^b - b^c}$.

11. $\frac{b^c + d^c}{b^2 + d^2 - bd}$.

12. $\frac{e^c - d^c}{e^2 + ed + d^2}$.

NOTE. In finding the value of a compound expression the operations indicated for each term must be performed *before* the operation indicated by the sign prefixed to the term. Indicated divisions should be written in the fractional form, and the sign $\times$ should be omitted between a figure and a letter, or between two letters. Thus, $(b - c) \div 2 \times c + 26$ should be written $\frac{b - c}{2c} + 26$.

Simplify the following expressions:

13. $100 + 80 \div 4$.

14. $75 - 25 \times 2$.

15. $25 + 5 \times 4 - 10 \div 5$.

16. $24 - 5 \times 4 \div 10 + 3$.

17. $(24 - 5) \times (4 \div 10 + 3)$.

If $a = 2$, $b = 10$, $x = 3$, $y = 5$, find the value of

18. $xy + 4a \times 2$.

19. $xy - 15b \div 5$.

20. $3x + 7y \div 7 + a \times y$.

21. $6b - 8y \div 2y \times b - 2b$.

22. $(6b-8y)\div 2y\times b+2b$.

23. $6b-(8y\div 2y)\times b-2b$.

24. $6b\div(b-y)-3x+bxy\div 10a$.

Exercise 2.

1. Express the sum of a and b.

2. Express the double of x.

3. By how much is a greater than 5?

4. If x is a whole number, find the next number above it.

5. Write five numbers in order of magnitude, so that x shall be the middle number.

6. What is the sum of $x+x+x+\dots\dots$ written a times?

7. If the product is xy and the multiplier x, what is the multiplicand?

8. A man who has a dollars spends b dollars; how many dollars has he left?

9. A regiment of men can be drawn up in a ranks of b men each, and there are c men over; of how many men does the regiment consist?

10. Write, the sum of x and y divided by c is equal to the product of a, b, and m, minus six times c, plus the quotient of a divided by the sum of x and y.

11. Write, six times the square of n, divided by m minus a, plus five b into the expression c plus d minus a.

12. Write, four times the fourth power of a, diminished by five times the square of a into the square of b; and increased by three times the fourth power of b.

CHAPTER II.

ADDITION AND SUBTRACTION.

INTEGRAL EXPRESSIONS.

24. If an algebraic expression contains only *integral forms*, that is, contains no *letter* in the denominator of any of its terms, it is called an **integral expression.** Thus, $x^3 + 7cx^2 - c^3 - 5c^2x$ and $\frac{1}{2}ax - \frac{1}{3}bcy$ are integral expressions, but $\dfrac{2c^2 - 4b - c - x}{a^2 - ab + b^2}$ is a **fractional expression.**

An integral expression may have for some values of the letters a fractional value, and a fractional expression an integral value. If, for instance, a stands for $\frac{3}{4}$ and b for $\frac{1}{4}$, the integral expression $2a - 5b$ stands for $\frac{6}{4} - \frac{5}{4} = \frac{1}{4}$; and the fractional expression $\dfrac{5a}{3b}$ stands for $\frac{15}{4} \div \frac{3}{4} = 5$. Integral and fractional expressions, therefore, are so named on account of the *form of the expressions*, and with no reference whatever to the numerical value of the expressions when definite numbers are put in place of the letters.

25. Definition of Addition. The process of finding the result when two or more numbers are taken together is called **addition,** and the result is called the **sum.**

26. Definition of Subtraction. The process of finding the result when one number is taken from another is called **subtraction,** and the result is called the **difference** or **remainder.**

The number taken away is called the **subtrahend;** the number from which the subtrahend is taken is called the **minuend.**

In practice the difference is found by discovering the number which must be *added* to the subtrahend to give the minuend. Hence the general definition of subtraction is

The operation of finding from two given numbers, called *minuend* and *subtrahend*, a third number, called *difference*, which *added to the subtrahend will give the minuend.*

27. Parentheses for Algebraic Numbers. An algebraic number which is to be added or subtracted is often inclosed in a parenthesis, in order that the signs + and −, which are used to distinguish positive and negative numbers, may not be confounded with the + and − signs that denote the operations of addition and subtraction. Thus $+4+(-3)$ expresses the sum, and $+4-(-3)$ expresses the difference, of the numbers $+4$ and -3.

28. Addition of Algebraic Numbers. In order to add two algebraic numbers, we begin at the place in the series which the first number occupies and count, *in the direction indicated by the sign of the second number*, as many units as there are in the absolute value of the second number.

The sum of $+4+(+3)$ is found by counting from $+4$ three units in *the positive direction;* that is, to the right, and is, therefore, $+7$.

The sum of $+4+(-3)$ is found by counting from $+4$ three units in *the negative direction;* that is, to the left, and is, therefore, $+1$.

$$\cdots\cdots -5 \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad +1 \quad +2 \quad +3 \quad +4 \quad +5 \quad +6 \cdots\cdots$$

The sum of $-4+(+3)$ is found by counting from -4 three units in the positive direction, and is, therefore, -1.

The sum of $-4+(-3)$ is found by counting from -4 three units in the negative direction, and is, therefore, -7.

Hence to add two or more algebraic numbers, we have the following rules:

I. If the numbers have *like* signs. *Find the sum of their absolute values, and prefix the common sign to the result.*

II. If there are two numbers with *unlike* signs. *Find the difference of their absolute values, and prefix to the result the sign of the greater number.*

III. If there are more than two numbers with *unlike* signs. *Combine the first two numbers and this result with the third number, and so on;* or, *find the sum of the positive numbers and the sum of the negative numbers, take the difference between the absolute values of these two sums, and prefix to the result the sign of the greater sum.*

29. The result in each case is called the **sum.** It is often called the **algebraic sum,** to distinguish it from the *arithmetical sum*, that is, the sum of the absolute values of the numbers.

30. Subtraction of Algebraic Numbers. In order to subtract one algebraic number from another, we begin at the place in the series which the minuend occupies and count, *in the direction opposite to that indicated by the sign of the subtrahend*, as many units as there are in the absolute value of the subtrahend.

Thus, the result of subtracting $+3$ from $+4$ is found by counting from $+4$ three units in the *negative direction;* that is, in the direction *opposite to that indicated by the sign* $+$ *before* 3, and is, therefore, $+1$.

The result of subtracting -3 from $+4$ is found by counting from $+4$ three units in the positive direction; that is, in the direction *opposite to that indicated by the sign* $-$ *before* 3, and is, therefore, $+7$.

$$\cdots\cdots -5 \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad +1 \quad +2 \quad +3 \quad +4 \quad +5 \quad +6 \cdots\cdots$$

The result of subtracting $+3$ from -4 is found by counting from -4 three units in the *negative direction*, and is, therefore, -7.

The result of subtracting -3 from -4 is found by counting from -4 three units in the *positive direction*, and is, therefore, -1.

Collecting the results obtained in addition and subtraction, we have

ADDITION.	SUBTRACTION.
$+4+(-3)=+4-3=+1.$	$+4-(+3)=+4-3=+1.$
$+4+(+3)=+4+3=+7.$	$+4-(-3)=+4+3=+7.$
$-4+(-3)=-4-3=-7.$	$-4-(+3)=-4-3=-7.$
$-4+(+3)=-4+3=-1.$	$-4-(-3)=-4+3=-1.$

If we employ the general symbols a and b to represent the absolute values of any two algebraic numbers, we have

ADDITION.	SUBTRACTION.	
$+a+(-b)=+a-b.$	$+a-(+b)=+a-b.$	(1)
$+a+(+b)=+a+b.$	$+a-(-b)=+a+b.$	(2)
$-a+(-b)=-a-b.$	$-a-(+b)=-a-b.$	(3)
$-a+(+b)=-a+b.$	$-a-(-b)=-a+b.$	(4)

From (1) and (3), it is seen that *subtracting a positive number is equivalent to adding an equal negative number.*

From (2) and (4), it is seen that *subtracting a negative number is equivalent to adding an equal positive number.*

Hence, to subtract one algebraic number from another:

Change the sign of the subtrahend, and add the subtrahend to the minuend.

This rule is consistent with the definition of subtraction given in § 26; for, if we have to subtract -4 from $+3$, we must add $+4$ to the subtrahend, -4, to cancel it, and then add $+3$ to obtain the minuend; that is, we must add $+7$ to the subtrahend to get the minuend, but $+7$ is obtained by changing the sign of the subtrahend, -4, making it $+4$, and adding it to $+3$, the minuend.

31. The Commutative Law of Addition. If we have a group of 3 things and another group of 4 things, we shall have a group of 7 things, whether we put the 3 things with the 4 things or the 4 things with the 3 things.

That is, $4 + 3 = 3 + 4.$

If now we have -3 to add to $+4$, we begin at $+4$ in the series, count three units to the left, and arrive at $+1$; and if we have $+4$ to add to -3, we begin at -3 in the series, count four units to the right, and arrive at $+1$.

That is, $+4 + (-3) = -3 + (+4).$

Hence, if a and b stand for *any* two numbers whatever, we have

$$a + b = b + a.$$

This is called the commutative law of addition, and may be stated as follows:

Additions may be performed in any order.

32. The Associative Law of Addition. If we have several numbers to be added, the result will evidently be the same, whether we add the numbers in succession or arrange them in groups and add the sums of these groups.

Thus,
$$\begin{aligned} & a + b + c + d + e \\ = & a + (b + c) + (d + e) \\ = & (a + b) + (c + d + e). \end{aligned}$$

This is called the associative law of addition, and may be stated as follows:

The terms of an expression may be grouped in any manner.

33. Addition of Integral Expressions. The addition of two integral expressions can be represented by connecting the second expression with the first by the sign $+$. If there are no like terms in the two expressions, the operation is *algebraically complete* when the two expressions are thus connected.

If, for example, it is required to add $m+n-p$ to $a+b+c$, the result will be $a+b+c+(m+n-p)$.

34. If, however, there are like terms in the expressions to be added, the like terms can be *collected*; that is, every set of like terms can be replaced by a single term with a coefficient equal to the algebraic sum of the coefficients of the like terms.

If it is required to add $5a^2+4a+3$ to $2a^2-3a-4$, the result will be

$$\begin{aligned} &2a^2-3a-4+(5a^2+4a+3) \\ &\quad=2a^2-3a-4+5a^2+4a+3 && \text{§ 32} \\ &\quad=2a^2+5a^2-3a+4a-4+3 && \text{§ 31} \\ &\quad=(2a^2+5a^2)+(-3a+4a)+(-4+3) && \text{§ 32} \\ &\quad=7a^2+a-1. \end{aligned}$$

This process is more conveniently represented by arranging the terms in columns, so that like terms shall stand in the same column, as follows:

$$\begin{array}{r} 2a^2-3a-4 \\ 5a^2+4a+3 \\ \hline 7a^2+a-1 \end{array}$$

The coefficient of a^2 in the result will be $5+2$, or 7; the coefficient of a will be $-3+4$, or 1; and the last term is $-4+3$, or -1.

If we are required to find the sum of

$2a^3 - 3a^2b + 4ab^2 + b^3$, $a^3 + 4a^2b - 7ab^2 - 2b^3$, $-3a^3 + a^2b - 3ab^2 - 4b^3$, and $2a^3 + 2a^2b + 6ab^2 - 3b^3$,

we write them in columns, as follows:

$$\begin{array}{rrrr} 2a^3 & -\,3a^2b & +\,4ab^2 & +\,b^3 \\ a^3 & +\,4a^2b & -\,7ab^2 & -\,2b^3 \\ -\,3a^3 & +\,a^2b & -\,3ab^2 & -\,4b^3 \\ 2a^3 & +\,2a^2b & +\,6ab^2 & -\,3b^3 \\ \hline 2a^3 & +\,4a^2b & & -\,8b^3 \end{array}$$

The coefficient of a^3 in the result will be $2 + 1 - 3 + 2$, or $+2$; the coefficient of a^2b will be $-3 + 4 + 1 + 2$, or $+4$; the coefficient of ab^2 will be $4 - 7 - 3 + 6$, or 0; and the coefficient of b^3 will be $1 - 2 - 4 - 3$, or -8.

Exercise 3.

Perform the additions indicated:

1. $(+16) + (-11)$.
2. $(-15) + (-25)$.
3. $(+68) + (-79)$.
4. $(-7) + (+4)$.
5. $(+33) + (+18)$.
6. $(+378) + (+709) + (-592)$.
7. A man has \$5242 and owes \$2758. How much is he worth?
8. The First Punic War began B.C. 264, and lasted 23 years. When did it end?
9. Augustus Cæsar was born B.C. 63, and lived 77 years. When did he die?
10. A man goes 65 steps forwards, then 37 steps backwards, then again 48 steps forwards. How many steps does he take in all? How many steps is he from where he started?

Exercise 4.

Perform the additions indicated:

1. $(+5ab)+(-5ab)$.
2. $(+8mx)+(-2mx)$.
3. $(-13mng)+(-7mng)$.
4. $(-5x^2)+(+8x^2)$.
5. $(+25my^2)+(-18my^2)$.
6. $(+7ab)+(-5ab)$.
7. $(+120my)+(-95my)$.
8. $(-33ab^2)+(+11ab^2)$.
9. $(-75xy)+(+20xy)$.
10. $(+15a^2x^2)+(-a^2x^2)$.
11. $(+5a)+(-3b)+(+4a)+(-7b)$.
12. $(+4a^2c)+(-10xyz)+(+6a^2c)+(-9xyz)$ $+(-11a^2c)+(+20xyz)$.
13. $(+3x^2y)+(-4ab)+(-2mn)+(+5x^2y)$ $+(-x^2y)+(-4x^2y)$.

Exercise 5.

Add the following expressions:

1. $5a+3b+c$, $3a+3b+3c$, $a+3b+5c$.
2. $7a-4b+c$, $6a+3b-5c$, $-12a+4c$.
3. $a+b-c$, $b+c-a$, $c+a-b$, $a+b-c$.
4. $a+2b+3c$, $2a-b-2c$, $b-a-c$, $c-a-b$.
5. $a-2b+3c+4d$, $3b-4c+5d-2a$, $5c-6d+3a-4b$, $7d-4a+5b-4c$.
6. x^3-4x^2+5x-3, $2x^3-7x^2-7x^2-14x+5$, $-x^3+9x^2+x+8$.
7. $x^4-2x^3+3x^2$, x^3+x^2+x, $4x^4+5x^3$, $2x^2+3x-4$, $-3x^2-2x-5$.
8. $a^3+3ab^2-3a^2b-b^3$, $2a^3+5a^2b-6ab^2-7b^2$, $a^3-ab^2+2b^3$.
9. $2ab-3ax^2+2a^2x$, $12ab-6a^2x+10ax^2$, $ax^3-8ab-5a^2x$.

10. $c^4 - 3c^3 + 2c^2 - 4c + 7,\ 2c^4 + 3c^3 + 2c^2 + 5c + 6,$
$-4c^4 - 4c^2 - 5.$

11. $3x^2 - xy + xz - 3y^2 + 4yz - z^2,\ -5x^2 - xy - xz + 5yz,$
$6x^2 - 6y - 6z,\ 4yz - 5yz + 3z^2,$
$-4x^2 + y^2 + 3yz + 3z^2.$

12. $m^5 - 3m^4n - 6m^3n^2,\ +m^3n^2 + m^2n^3 - 5m^4n,$
$7m^3n^2 + 4m^2n^3 - 3mn^4,\ -2m^2n^3 - 3mn^4 + 4n^5,$
$2mn^4 + 2n^5 + 3m^5,\ -n^5 + 2m^5 + 7m^4n.$

Exercise 6.

Perform the subtractions indicated (§ 30):

1. $(+25) - (+16)$.
2. $(-50) - (-25)$.
3. $(-31) - (+58)$.
4. $(+107) - (-93)$.
5. Rome was ruled by emperors from B.C. 30, to its fall, A.D. 476. How long did the empire last?
6. The continent of Europe lies between 36° and 71° north latitude, and between 12° west and 63° east longitude (from Paris). How many degrees does it extend in latitude, and how many in longitude?

Exercise 7.

Perform the operations indicated:

1. $(+5x) - (-4x)$.
2. $(-3ab) - (+5ab)$.
3. $(+3ab^2) - (+10ab^2)$.
4. $(+15m^2x^2) - (-7m^2x^2)$.
5. $(-7ay) - (-3ay)$.
6. $(+17ax^3) - (-24ax^3)$.
7. $(+5a^2x) - (-3a^2x)$.
8. $(-4xy) - (-5xy)$.
9. $(+8ax) - (-3ay)$.
10. $(+2ab^2y) - (+aby)$.
11. $(+9x^2) + (5x^2) - (+8x^2)$.
12. $(+5x^2y) - (-18x^2y) + (-10x^2y)$.

13. $(+17ax^3)-(-ax^3)-(+24ax^3)$.

14. $(-3ab)+(2mx)-(-4mx)$.

15. $(+3a)-(+2b)-(-4c)$.

Exercise 8.

In performing the following subtractions change the signs of the subtrahend **mentally** and add:

1. From $6a-2b-c$ take $2a-2b-3c$.
2. From $3a-2b+3c$ take $2a-7b-c-b$.
3. From $7x^2-8x-1$ take $5x^2-6x+3$.
4. From $4x^4-3x^3-2x^2-7x+9$
   take $x^4-2x^3-2x^2+7x-9$.
5. From $2x^2-2ax+3a^2$ take x^2-ax+a^2.
6. From $x^2-3xy-y^2+yz-2z^2$
   take $x^2+2xy+5xz-3y^2-2z^2$.
7. From $a^3-3a^2b+3ab^2-b^3$
   take $-a^3+3a^2b-3ab^2+b^3$.
8. From $x^2-5xy+xz-y^2+7yz+2z^2$
   take $x^2-xy-xz+2yz+3z^2$.
9. From $2ax^2+3abx-4b^2x+12b^3$
   take $ax^2-4abx+bx^2-5b^2x-x^3$.
10. From $6x^3-7x^2y+4xy^2-2y^3-5x^2+xy-4y^2+2$
    take $8x^3-7x^2y+xy^2-y^3+9x^2-xy+6y^2-4$.
11. From a^4-b^4 take $4a^3b-6a^2b^2+4ab^3$, and from the result take $2a^4-4a^3b+6a^2b^2+4ab^3-2b^4$.
12. From $x^3y^2-3x^2y^3+4xy^4-y^5$ take $-x^5+2x^4y-4xy^4-4y^5$. Add the same two expressions, and subtract their difference from their sum.
13. From $a^2b^2-a^2bc-8ab^2c-a^2c^2+abc^2-6b^2c^2$
    take $2a^2bc-5ab^2c+2abc^2-5b^2c^2$.

14. From $12a+3b-5c-2d$ take $10a-b+4c-3d$, and show that the result is numerically correct when $a=6$, $b=4$, $c=1$, $d=5$.

15. What number must be added to a to make b; and what number must be taken from $2a^3-6a^2b+6ab^2-2b^3$ to leave $a^3-7a^2b-3b^3$?

16. From $2x^2-y^2-2xy+z^2$ take $x^2-y^2+2xy-z^2$.

17. From $12ac+8cd-9$ take $-7ac-9cd+8$.

18. From $-6a^2+2ab-3c^2$ take $4a^2+6ab-4c^2$.

19. From $9xy-4x-3y+7$ take $8xy-2x+3y+6$.

20. From $-a^2bc-ab^2c+abc^2-abc$ take $a^2bc+ab^2c-abc^2+abc$.

21. From $7x^2-2x+4$ take $2x^2+3x-1$.

22. From $3x^2+2xy-y^2$ take $-x^2-3xy+3y^2$, and from the remainder take $3x^2+4xy-5y^2$.

23. From ax^2-by^2 take cx^2-dy^2.

24. From $ax+bx+by+cy$ take $ax-bx-by+cy$.

25. From $5x^2+4x-4y+3y^2$ take $5x^2-3x+3y+y^2$.

26. From $a^2b^2+12abc-9ax^2$ take $4ab^2-6acx+3a^2x$.

27. From $a^2-2ab+c^2-3b^2$ take $2a^2-2ab+3b^2$.

28. From the sum of the first four of the following expressions, $a^2+b^2+c^2+d^2$, $d^2+b^2+c^2$, $a^2-c^2+b^2-d^2$, $a^2-b^2+c^2+d^2$, $b^2+c^2+d^2-a^2$, take the sum of the last four.

29. From $2x^2-2y^2-z^2$ take $3y^2+2x^2-z^2$, and from the remainder take $3z^2-2y^2-x^2$.

30. From $a^3-2a^2c+3ac^2$ take the sum of $a^2c-2a^3+2ac^2$ and $a^3-ac^2-a^2c$.

35. Rules for removing Parentheses. From (§ 30), it appears that

$$a+(+b)=a+b. \qquad a-(+b)=a-b.$$
$$a+(-b)=a-b. \qquad a-(-b)=a+b.$$

The same rules for removing parentheses hold true whether one or more terms are inclosed. Hence, when an expression within a parenthesis is preceded by a **plus sign,** the parenthesis may be removed.

When an expression within a parenthesis is preceded by a **minus sign,** the parenthesis may be removed *if the sign of every term within the parenthesis is changed.*

Thus, $$a+(b-c)=a+b-c.$$
$$a-(b-c)=a-b+c.$$

36. Expressions may occur with more than one parenthesis. In such cases parentheses of different shapes are used, and the beginner when he meets with a (or a [or a { must look carefully for the other part, whatever may intervene; and all that is included between the two parts of each parenthesis must be treated as the sign before it directs, without regard to other parentheses. It is best to remove each parenthesis in succession, *beginning with the innermost one.* Thus,

(1) $$\begin{aligned} &5a-\{-3a-[3a-(2a-\overline{a-b})-a]+a\} \\ &=5a-\{-3a-[3a-(2a-a+b)-a]+a\} \\ &=5a-\{-3a-[3a-2a+a-b-a]+a\} \\ &=5a-\{-3a-3a+2a-a+b+a+a\} \\ &=5a+3a+3a-2a+a-b-a-a \\ &=5a+3a+3a-2a+a-a-a-b \\ &=8a-b. \end{aligned}$$

NOTE. The sign $-$ which is written in the above problem before the first term a under the vinculum is really the sign of the vinculum, $-\overline{a-b}$ meaning the same as $-(a-b)$.

Exercise 9.

Simplify the following expressions by removing the parentheses and collecting like terms:

1. $(a+b)+(b+c)-(a+c)$.
2. $(2a-b-c)-(a-2b+c)$.
3. $(2x-y)-(2y-z)-(2z-x)$.
4. $(a-x-y)-(b-x+y)+(c+2y)$.
5. $(2x-y+3z)+(-x-y-4z)-(3x-2y-z)$.
6. $(3a-b+7c)-(2a+3b)-(5b-4c)+(3c-a)$.
7. $1-(1-a)+(1-a+a^2)-(1-a+a^2-a^3)$.
8. $a-\{2b-(3c+2b)-a\}$.
9. $2a-\{b-(a-2b)\}$.
10. $3a-\{b+(2a-b)-(a-b)\}$.
11. $7a-[3a-\{4a-(5a-2a)\}]$.
12. $2x+(y-3z)-\{(3x-2y)+z\}+5x-(4y-3z)$.
13. $\{(3a-2b)+(4c-a)\}-\{a-(2b-3a)-c\}$ $+\{a-(b-5c-a)\}$.
14. $a-[2a+(3a-4a)]-5a-\{6a-[(7a+8a)-9a]\}$.
15. $2a-(3b+2c)-[5b-(6c-6b)+5c$ $-\{2a-(c+2b)\}]$.
16. $a-[2b+\{3c-3a-(a+b)\}+\{2a-(b+c)\}]$.
17. $16-x-[7x-\{8x-(9x-\overline{3x-6x})\}]$.
18. $2a-[3b+(2b-c)-4c+\{2a-(3b-\overline{c-2b})\}]$.
19. $a-[2b+\{3c-3a-(a+b)\}+2a-(b+3c)]$.
20. $a-[5b-\{a-(3c-3b)+2c-(a-2b-c)\}]$.

37. Rules for Introducing Parentheses. The rules for introducing parentheses follow directly from the rules for removing them:

1. Any number of terms of an expression may be put within a parenthesis, and the sign **plus** placed before it.

2. Any number of terms of an expression may be put within a parenthesis, and the sign **minus** placed before the parenthesis; *provided the sign of every term within the parenthesis is changed.*

It is usual to prefix to the parenthesis the sign of the first term that is to be inclosed within it.

Exercise 10.

Express in binomials, and also in trinomials:

1. $2a - 3b - 4c + d + 3e - 2f.$

2. $a - 2x + 4y - 3z - 2b + c.$

3. $a^5 + 3a^4 - 2a^3 - 4a^2 + a - 1.$

4. $-3a - 2b + 2c - 5d - e - 2f.$

5. $ax - by - cz - bx + cy + az.$

6. $2x^5 - 3x^4y + 4x^3y^2 - 5x^2y^3 + xy^4 - 2y^5.$

7. Express each of the above in trinomials, each trinomial having its last two terms inclosed by *inner* parentheses.

Collect in parentheses the coefficients of x, y, z in

8. $2ax - 6ay + 4bz - 4bx - 2cx - 3cy.$

9. $ax - bx + 2ay + 3y + 4az - 3bz - 2z.$

10. $ax - 2by + 5cz - 4bx - 3cy + az - 2cx - ay + 4bz.$

11. $12ax + 12ay + 4by - 12bz - 15cx + 6cy + 3cz.$

12. $2ax - 3by - 7cz - 2bx + 2cx + 8cz - 2cx - cy - cz.$

CHAPTER III.

MULTIPLICATION

Integral Expressions.

38. Definition of Multiplication. The process of finding the result when a given number is taken as many times as there are units in another number is called **multiplication,** and the result is called the **product.**

This definition fails when the multiplier is a fraction, for we cannot take the multiplicand a *fraction of a time.* We therefore consider what extension of the meaning of multiplication can be made so as to cover the case in question. When we multiply by a fraction, we divide the multiplicand into as many equal parts as there are units in the denominator and take as many of these parts as there are units in the numerator. If, for instance, we multiply 8 by $\frac{3}{4}$, we divide 8 into four equal parts and take three of these parts, getting 6 for the product. We see that $\frac{3}{4}$ is $\frac{3}{4}$ of 1, and 6 is $\frac{3}{4}$ of 8; that is, the product 6 is obtained from the multiplicand 8 precisely as the multiplier $\frac{3}{4}$ is obtained from 1.

Again, in $6 \times 8 = 48$,

the multiplier $\quad 6$ is $1 + 1 + 1 + 1 + 1 + 1$,

and the product $\quad 48$ is $8 + 8 + 8 + 8 + 8 + 8$.

Hence we have for the general definition of multiplication,

The operation of finding from two given numbers, called *multiplicand* and *multiplier*, a third number called *product*,

which is *formed from the multiplicand as the multiplier is formed from unity.*

39. Law of Signs in Multiplication. By the definition of multiplication,

since $\qquad +3=(+1)+(+1)+(+1),$

$$\therefore\ 3\times(+4)=(+4)+(+4)+(+4)$$
$$=+12,$$

and $\qquad 3\times(-4)=(-4)+(-4)+(-4)$

$$=-12.$$

Again, since $\qquad 3=(-1)+(-1)+(-1),$

$$\therefore\ (-3)\times 4=(-4)+(-4)+(-4)$$
$$=-12,$$

and $\qquad (-3)\times(-4)=-(-4)-(-4)-(-4)$

$$=+4+4+4$$
$$=+12.$$

If we use a to represent the absolute value of any number, and b to represent the absolute value of any other number, we shall have

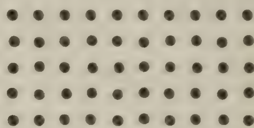
$$(+a)\times(+b)=+ab. \qquad (1)$$
$$(+a)\times(-b)=-ab. \qquad (2)$$
$$(-a)\times(+b)=-ab. \qquad (3)$$
$$(-a)\times(-b)=+ab. \qquad (4)$$

Hence, in finding the product of two algebraic numbers,

Like signs give $+$, *and unlike signs give* $-$.

40. It will be seen from the above cases that the *absolute value* of the product is *independent of the signs of the factors*, and that the *sign* of the product is *independent of the order of the factors.*

41. The Commutative Law of Multiplication. If we have five lines of dots with ten dots in a line, the whole number of dots will be expressed by 5×10.



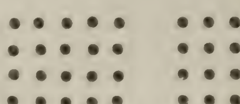
If we consider the dots as ten columns with five dots in a column, the number will be expressed by 10×5.

That is, $$5 \times 10 = 10 \times 5.$$

Hence, if a and b stand for any two positive numbers,

$$ab = ba.$$

42. The Distributive Law of Multiplication. The expression $4 \times (5 + 3)$ means that we are to take the sum of the numbers 5 and 3 four times. The process can be represented by placing five dots in a line, and a little to the right three more dots in the same line, and then placing a second, third, and fourth line of dots underneath the first line and exactly similar to it.

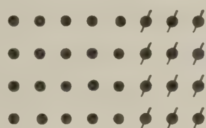


There are $(5 + 3)$ dots in each line, and 4 lines. The total number of dots, therefore, is $4 \times (5 + 3)$.

In the left-hand group there are 4×5 dots, and in the right-hand group 4×3 dots. The sum of these two numbers $(4 \times 5) + (4 \times 3)$ must be equal to the total number; that is,

$$4 \times (5 + 3) = (4 \times 5) + (4 \times 3). \qquad (1)$$

Again, the expression $4 \times (8 - 3)$ means that 3 is to be taken from 8, and the remainder to be multiplied by 4. The process can be represented by placing eight dots in a line and crossing the last three, and then placing a second, third, and fourth line of dots underneath the first line and exactly similar to it.



The whole number of dots not crossed in each line is evidently $(8 - 3)$, and the whole number of lines is 4. Therefore the total number of dots not crossed is

$$4 \times (8 - 3).$$

The total number of dots (crossed and not crossed) is (4×8), and the total number of dots crossed is (4×3). Therefore the total number of dots not crossed is

$$(4 \times 8) - (4 \times 3).$$

Hence, $$4 \times (8 - 3) = (4 \times 8) - (4 \times 3). \qquad (2)$$

If a, b, and c stand for any positive numbers, (1) becomes

$$a \times (b + c) = ab + ac,$$

and (2) becomes

$$a \times (b - c) = ab - ac.$$

This is the distributive law of multiplication and may be stated as follows:

In multiplying a compound expression by a simple expression, the result is obtained by multiplying each term of the compound expression by the simple expression, and writing down the successive products with the same signs as those of the original terms.

43. The Associative Law of Multiplication. The product of three or more factors is evidently the same in whatever way the factors are grouped. Thus,

$$7 \times (3 \times 5) = 3 \times (5 \times 7) = 5 \times (7 \times 3) = 105.$$

Hence, $c \times (a \times b) = a \times (b \times c) = b \times (c \times a) = abc$,

where a, b, and c stand for any positive numbers.

This is called the associative law of multiplication, and may be stated as follows:

The factors of a product may be grouped in any manner.

44. Since (§ 40) the absolute value of a product is independent of the signs of its factors, and the sign of a product is independent of the order of its factors, it is evident that the commutative, the distributive, and the associative laws of multiplication apply to algebraic as well as to arithmetical numbers.

45. The Index Law of Multiplication.

Since $a^2 = aa$, and $a^3 = aaa$, § 19

$$a^2 \times a^3 = aa \times aaa = a^5 = a^{2+3};$$

$$a^4 \times a = aaaa \times a = a^5 = a^{4+1}.$$

If m and n stand for any *positive integers*,

since $a^m = aaa \cdots\cdots$ to m factors,

and $a^n = aaa \cdots\cdots$ to n factors,

$$\begin{aligned} a^m \times a^n &= (aaa \cdots\cdots \text{ to } m \text{ factors}) \times (aaa \cdots\cdots \text{ to } n \text{ factors}) \\ &= aaa \cdots\cdots \text{ to } (m+n) \text{ factors}, \\ &= a^{m+n}. \end{aligned}$$

Hence,

The index of the product of two powers of the same number is equal to the sum of the indices of the factors.

46. The commutative, the distributive, the associative, the index laws, and the *law of signs*, constitute the fundamental laws of Algebra.

47. Multiplication of Monomials. When the factors are single letters, the product is represented by simply writing the letters without any sign between them.

Thus, the product of a, b, and c is expressed by abc; and the product of $4a$, $5b$, and $3c$ is

$$4a \times 5b \times 3c = 4 \times 5 \times 3abc = 60abc. \qquad \S\ 41$$

NOTE. We cannot write 453 for $4 \times 5 \times 3$, because another meaning has been assigned in Arithmetic to 453; namely, $400 + 50 + 3$. Hence between arithmetical factors the sign $\times$ must be written.

48. By the law of signs, we have

$$(-a) \times (-b) = +ab,$$

and $$(+ab) \times (-c) = -abc;$$

that is, $$(-a) \times (-b) \times (-c) = -abc.$$

Hence, the product of an *even* number of negative factors will be *positive*, and the product of an *odd* number of negative factors will be *negative*.

49. The product of a^2b and a^3b^2 is

$$a^2b \times a^3b^2 = a^2a^3bb^2 = a^{2+3}b^{1+2} = a^5b^3.$$

50. To multiply one monomial by another, therefore,

Find the product of the coefficients, and to this product annex the letters, giving to each letter in the product an index equal to the sum of its indices in the factors.

51. $(abc)^2$ means $abc \times abc$, which equals $aabbcc$, or $a^2b^2c^2$. In like manner, $(abc)^n = a^nb^nc^n$. That is,

The nth power of the product of several factors is equal to the product of the nth powers of the factors.

52. Polynomials by Monomials. We have (§ 42),

$$a(b+c)=ab+ac;$$

and,
$$a(b-c+d-e)=ab-ac+ad-ae.$$

To multiply a polynomial by a monomial, therefore,

Multiply each term of the polynomial by the monomial, and add the partial products.

Exercise 11.

Find the product of

1. -17 and 8.
2. -12.8 and 25.
3. 3.29 and 5.49.
4. -18 and -5.
5. 43 and -6.
6. 457 and 100.
7. $(-358-417)$ and -79.
8. $(7.512-\{-2.894\})$ and $(-6.037+\{13.963\})$.
9. 13, 8, and -7.
10. -38, 9, and -6.
11. -20.9, -1.1, and 8.
12. -78.3, -0.57, $+1.38$, and -27.9.
13. -2.906, -2.076, -1.49, and 0.89.

Exercise 12.

Find the product of

1. $6a$ and $-2a$.
2. $5mn$ and $9m$.
3. $3ax$ and $-4by$.
4. $-8cm$ and dn.

5. $-7ab$ and $2ac$.

6. $5m^2x$ and $3mx^2$.

7. $5a^m$ and $-2a^n$.

8. $3a^2x^2$ and $7a^3x^4$.

9. $7a$, $-4b$, and $-8c$.

10. $8ab^2$, $3ac$, and $-4c^2$.

11. $27ab$, $-39mp$, and $18ap$.

12. $6ab^2y^3$, $2b^3y^3$, and $-5a^2y$.

13. $7m^2x$, $3mx^2$, and $-2mq$.

14. $-3pq^2$, $6p^3q$, and $8p^2q^3$.

15. $2a^2m^3x^4$, $3am^5x^2$, and $4a^3mx^2$.

16. $6x^3yz^3$, $-9x^2y^2z^2$, and $3x^4yz$.

17. $3ax$, $2am$, $-4mx$, and b^2.

18. $7am^2$, $3b^2n^2$, $-4ab$, a^2bn, $-2b^2n$, and $-mn$.

19. $2ab^2$, $-5a^2b$, $-3ab$, and $7a$.

Exercise 13.

Find the product of

1. $(4a^2-3b)$ and $3ab$.

2. $(8a^2-9ab)$ and $3a^2$.

3. $(3x^2-4y^2+5z^2)$ and $2x^2y$.

4. $(a^3x-5a^2x^2+ax^3+2x^4)$ and ax^2y.

5. $(-9a^5+3a^3b^2-4a^2b^3-b^5)$ and $-3ab^4$.

6. $(3x^3-2x^2y-7xy^2+y^3)$ and $-5x^2y$.

7. $(-4xy^2+5x^2y+8x^3)$ and $-3x^2y$.

8. $(-3+2ab+a^2b^2)$ and $-a^4$.

9. $(-z-2xz^2+5x^2yz^2-6x^3y^2+3x^3y^2z)$ and $-3x^3yz$.

53. Polynomials by Polynomials. If we have $m+n+p$ to be multiplied by $a+b+c$, we may substitute M for the multiplicand $m+n+p$. Then

$$(a+b+c)M = aM + bM + cM. \qquad \S\ 42$$

If now we substitute for M its value $m+n+p$, we shall have

$$\begin{aligned} &a(m+n+p)+b(m+n+p)+c(m+n+p)\\ &\quad = am+an+ap+bm+bn+bp+cm+cn+cp. \end{aligned}$$

That is, to find the product of two polynomials,

Multiply every term of the multiplicand by each term of the multiplier, and add the partial products.

54. In multiplying polynomials, it is a convenient arrangement to write the multiplier under the multiplicand, and place like terms of the partial products in columns.

(1)

$$\begin{array}{rrr} 5a & -\ 6\,b & \\ 3a & -\ 4\,b & \\ \hline 15a^2 & -\ 18ab & \\ & -\ 20ab & +\ 24b^2 \\ \hline 15a^2 & -\ 38ab & +\ 24b^2 \end{array}$$

We multiply $5a$, the first term of the multiplicand, by $3a$, the first term of the multiplier, and obtain $15a^2$; then $-6b$, the second term of the multiplicand, by $3a$, and obtain $-18ab$. The first line of partial products is $15a^2-18ab$. In multiplying by $-4b$, we obtain for a second line of partial products $-20ab+24b^2$, which is put one place to the right, so that the like terms $-18ab$ and $-20ab$ may stand in the same column. We then add the coefficients of the like terms, and obtain the complete product in its simplest form.

(2) Multiply $4x+3+5x^2-6x^3$ by $4-6x^2-5x$.

Arrange both multiplicand and multiplier according to the ascending powers of x.

$$
\begin{array}{llllll}
3+ & 4x+ & 5x^2- & 6x^3 & & \\
4- & 5x- & 6x^2 & & & \\
\hline
12+ & 16x+ & 20x^2- & 24x^3 & & \\
 & -15x- & 20x^2- & 25x^3+ & 30x^4 & \\
 & & -18x^2- & 24x^3- & 30x^4+ & 36x^5 \\
\hline
12+ & x- & 18x^2- & 73x^3 & & +36x^5
\end{array}
$$

(3) Multiply $1+2x+x^4-3x^2$ by x^3-2-2x.

Arrange according to the descending powers of x.

$$
\begin{array}{llllllll}
x^4 & -3x^2 & +2x & +1 & & & & \\
x^3 & -2x & -2 & & & & & \\
\hline
x^7 & -3x^5 & +2x^4 & +x^3 & & & & \\
 & -2x^5 & & +6x^3 & -4x^2 & -2x & & \\
 & & -2x^4 & & +6x^2 & -4x & -2 & \\
\hline
x^7 & -5x^5 & & +7x^3 & +2x^2 & -6x & -2 &
\end{array}
$$

(4) Multiply $a^2+b^2+c^2-ab-bc-ac$ by $a+b+c$.

Arrange according to descending powers of a.

$$
\begin{array}{llllllllll}
a^2 & -ab & -ac & +b^2 & -bc & +c^2 & & & & \\
a & +b & +c & & & & & & & \\
\hline
a^3 & -a^2b & -a^2c & +ab^2 & -abc & +ac^2 & & & & \\
 & +a^2b & & -ab^2 & -abc & & +b^3 & -b^2c & +bc^2 & \\
 & & +a^2c & & -abc & -ac^2 & & +b^2c & -bc^2 & +c^3 \\
\hline
a^3 & & & & -3abc & & +b^3 & & & +c^3
\end{array}
$$

NOTE. The student should observe that, with a view to bringing like terms of the partial products in columns, the terms of the multiplicand and multiplier are arranged in the *same order*.

Exercise 14.

Multiply:

1. $x^2 - 4$ by $x^2 + 5$.
2. $y - 6$ by $y + 13$.
3. $a^4 + a^2x^2 + x^4$ by $a^2 - x^2$.
4. $x^2 + xy + y^2$ by $x - y$.
5. $2x - y$ by $x + 2y$.
6. $2x^3 + 4x^2 + 8x + 16$ by $3x - 6$.
7. $x^3 + x^2 + x - 1$ by $x - 1$.
8. $x^2 - 3ax$ by $x + 3a$.
9. $2b^2 + 3ab - a^2$ by $-5b + 7a$.
10. $2a + b$ by $a + 2b$.
11. $a^2 + ab + b^2$ by $a - b$.
12. $a^2 - ab + b^2$ by $a + b$.
13. $2ab - 5b^2$ by $3a^2 - 4ab$.
14. $-a^3 + 2a^2b - b^3$ by $4a^2 + 8ab$.
15. $a^2 + ab + b^2$ by $a^2 - ab + b^2$.
16. $a^3 - 3a^2b + 3ab^2 - b^3$ by $a^2 - 2ab + b^2$.
17. $x + 2y - 3z$ by $x - 2y + 3z$.
18. $2x^2 + 3xy + 4y^2$ by $3x^2 - 4xy + yz$.
19. $x^2 + xy + y^2$ by $x^2 + xz + z^2$.
20. $a^2 + b^2 + c^2 - ab - ac - bc$ by $a + b + c$.
21. $x^2 - xy + y^2 + x + y + 1$ by $x + y - 1$.

Arrange the multiplicand and multiplier according to the descending powers of a common letter, and multiply:

22. $5x + 4x^2 + x^3 - 24$ by $x^2 + 11 - 4x$.
23. $x^3 + 11x - 4x^2 - 24$ by $x^2 + 5 + 4x$.
24. $x^4 + x^2 - 4x - 11 + 2x^3$ by $x^2 - 2x + 3$.
25. $-5x^4 - x^2 - x + x^5 + 13x^3$ by $x^2 - 2 - 2x$.
26. $3x + x^3 - 2x^2 - 4$ by $2x + 4x^3 + 3x^2 + 1$.

27. $5a^4 + 2a^2b^2 + ab^3 - 3a^3b$ by $5a^3b - 2ab^3 + 3a^2b^2 + b^4$.

28. $4a^7y - 32ay^4 - 8a^5y^2 + 16a^3y^3$ by $a^6y^2 + 4a^2y^4 + 4a^4y^3$.

29. $3m^3 + 3n^3 + 9mn^2 + 9m^2n$ by $6m^2n^3 - 2mn^4 - 6m^3n^2 + 2m^4n$.

30. $6a^5b + 3a^2b^4 - 2ab^5 + b^6$ by $4a^4 - 2ab^3 - 3b^4$.

Find the product of:

31. $x - 3$, $x - 1$, $x + 1$, and $x + 3$.

32. $x^2 - x + 1$, $x^2 + x + 1$, and $x^4 - x^2 + 1$.

33. $a^2 + ab + b^2$, $a^2 - ab + b^2$, and $a^4 - a^2b^2 + b^4$.

34. $4a^3 - 4a^2b + ab^2$, $4a^2 + 3ab + b^2$, and $2a^2b + b^3$.

35. $x + a$, $x + 2a$, $x - 3a$, $x - 4a$, and $x + 5a$.

36. $9a^2 + b^2$, $27a^3 - b^3$, $27a^3 + b^3$, and $81a^4 - 9a^2b^2 + b^4$.

37. From the product of $y^2 - 2yz - z^2$ and $y^2 + 2yz - z^2$ take the product of $y^2 - yz - 2z^2$ and $y^2 + yz - 2z^2$.

38. Find the dividend when the divisor $= 3a^2 - ab - 3b^2$, the quotient $= a^2b - 2b^2$, the remainder $= -2ab^4 - 6b^5$.

The multiplication of polynomials may be *indicated* by inclosing each in a parenthesis and writing them one after the other. When the operations indicated are actually performed, the expression is said to be *simplified.*

Simplify:

39. $(a + b - c)(a + c - b)(b + c - a)(a + b + c)$.

40. $(a + b)(b + c) - (c + d)(d + a) - (a + c)(b - d)$.

41. $(a + b + c + d)^2 + (a - b - c + d)^2 + (a - b + c - d)^2 + (a + b - c - d)^2$.

42. $(a+b+c)^2-a(b+c-a)-b(a+c-b)-c(a+b-c).$

43. $(a-b)x-(b+c)a-\{(b-x)(b-a)-(b-c)(b+c)\}.$

44. $(m+n)m-\{(m-n)^2-(n-m)n\}.$

45. $(a-b+c)^2-\{a(c-a-b)-[b(a+b+c)$
$-c(a-b-c)]\}.$

46. $(p^2+q^2)r-(p+q)(p\{r-q\}-q\{r-p\}).$

47. $(9x^2y^2-4y^4)(x^2-y^2)-\{3xy-2y^2\}\{3x(x^2+y^2)$
$-2y(y^2+3xy-x^2)\}y.$

48. $a^2-\{2ab-[-(a+\{b-c\})(a-\{b-c\})+2ab]$
$-4bc\}-(b+c)^2.$

49. $\{ac-(a-b)(b+c)\}-b\{b-(a-c)\}.$

50. $5\{(a-b)x-cy\}-2\{a(x-y)-bx\}$
$-\{3ax-(5c-2a)y\}.$

51. $(x-1)(x-2)-3x(x+3)+2\{(x+2)(x+1)-3\}.$

52. $\{(2a+b)^2+(a-2b)^2\}\times\{(3a-2b)^2-(2a-3b)^2\}.$

53. $4(a-3b)(a+3b)-2(a-6b)^2-2(a^2+6b^2).$

54. $x^2(x^2+y^2)^2-2x^2y^2(x+y)(x-y)-(x^3-y^3)^2.$

55. $16(a^2+b^2)(a^2-b^2)-(2a-3)(2a+3)(4a^2+9)$
$+(2b-3)(2b+3)(4b^2+9).$

CHAPTER IV.

DIVISION.

Integral Expressions.

55. Definition of Division. In division the *product* and *one factor* are given, and *the other factor* is required. We may therefore take for the general definition of division

The operation by which when *the product* and *one factor* are given *the other factor* is found.

With reference to this operation the product is called the **dividend**, the given factor the **divisor**, and the required factor the **quotient**.

56. Law of Signs for Division.

Since $(+a) \times (+b) = +ab$, $\therefore +ab \div (+a) = +b$.

Since $(+a) \times (-b) = -ab$, $\therefore -ab \div (+a) = -b$.

Since $(-a) \times (+b) = -ab$, $\therefore -ab \div (-a) = +b$.

Since $(-a) \times (-b) = +ab$, $\therefore +ab \div (-a) = -b$.

That is, if the dividend and divisor have like signs, the quotient has the sign $+$; and if they have unlike signs, the quotient has the sign $-$. Hence, in division,

Like signs give $+$; *unlike signs give* $-$.

57. Index Law for Division. If we have to divide a^5 by a^2, a^6 by a^4, a^4 by a, a^2 by a^5, we write them as follows:

$$\frac{a^5}{a^2} = \frac{aaaaa}{aa} = aaa = a^3 = a^{5-2};$$

$$\frac{a^6}{a^4} = \frac{aaaaaa}{aaaa} = aa = a^2 = a^{6-4};$$

$$\frac{a^4}{a} = \frac{aaaa}{a} = aaa = a^3 = a^{4-1};$$

$$\frac{a^2}{a^5} = \frac{aa}{aaaaa} = \frac{1}{aaa} = \frac{1}{a^3} = \frac{1}{a^{5-2}}.$$

58. To divide one monomial by another, therefore,

Write the dividend over the divisor with a line between them: if the expressions have common factors, remove the common factors.

Thus, $\frac{4x^2y}{2x} = 2xy$; $\frac{14a^2b}{2a} = 7ab$; $\frac{54x^5y^3z^2}{-6x^3y^2z} = -9x^2yz$;

$$\frac{3a^{4n-1}}{3a^{4n-1}} = 1;\quad \frac{12x^{p-4}y^{r+3}}{3x^{p-6}y^{r-1}} = 4x^2y^4.$$

In the last example, $(p-4)-(p-6)=p-4-p+6=2$ and $(r+3)-(r-1)=r+3-r+1=4$.

NOTE. Since $a^n \div a^n = 1$, and by the rule $= a^{n-n} = a^0$, it follows that $a^0 = 1$. Hence, any *letter* in the quotient *with zero for an index* may be omitted without affecting the quotient.

59. To divide a polynomial by a monomial, we have, by the *distributive law*, the following rule:

Divide each term of the dividend by the divisor, and add the partial quotients.

Thus, $\frac{8ab + 4ac - 6ad}{2a} = \frac{8ab}{2a} + \frac{4ac}{2a} - \frac{6ad}{2a}$

$$= 4b + 2c - 3d.$$

$$\frac{9a^4b^2x - 12a^3bx^2 - 3a^2x}{3a^2x} = \frac{9a^4b^2x}{3a^2x} - \frac{12a^3bx^2}{3a^2x} - \frac{3a^2x}{3a^2x}$$

$$= 3a^2b^2 - 4abx - 1.$$

$$\frac{6x^{4n+1} - 4x^{3n}}{2x^{2n-1}} = \frac{6x^{4n+1}}{2x^{2n-1}} - \frac{4x^{3n}}{2x^{2n-1}} = 3x^{2n+2} - 2x^{n+1}.$$

NOTE. Here we have $4n+1-(2n-1)=4n+1-2n+1=2n+2$, and $3n-(2n-1)=3n-2n+1=n+1$, as indices of x in the first and last terms of the quotient respectively.

Exercise 15.

Perform the operations indicated :

1. $\frac{+264}{+4}$.
2. $\frac{-3840}{-8}$.
3. $\frac{+3840}{+30}$.
4. $\frac{-2568}{+12}$.
5. $\frac{106.33}{-4.9}$.
6. $\frac{-42.435}{+34.5}$.
7. $\frac{-264}{+24}$.
8. $\frac{-3670}{-85}$.
9. $\frac{+6.8503}{-61}$.
10. $\frac{-7.1560}{+324}$.
11. $\frac{-1}{-3.14159}$.
12. $\frac{-0.31831}{-31.4159}$.

Exercise 16.

Perform the operations indicated :

1. $\frac{+ab}{+a}$.
2. $\frac{+ab}{-a}$.
3. $\frac{-ab}{+a}$.
4. $\frac{-ab}{-a}$.
5. $\frac{6mx}{2x}$.
6. $\frac{12a^4}{-3a}$.
7. $\frac{10ab}{2bc}$.
8. $\frac{x^3}{-x^5}$.
9. $\frac{-12am}{-2m}$.
10. $\frac{35abc}{5bd}$.
11. $\frac{abx}{5aby}$.
12. $\frac{27a^7}{-3a^3}$.
13. $\frac{-3bmx}{4ax^2}$.
14. $\frac{ab^2c^3}{abc}$.
15. $\frac{m^5p^2x^4}{mp^2x^2}$.
16. $\frac{-51abdy^2}{3bdy}$.
17. $\frac{225m^2y}{25my^2}$.
18. $\frac{30x^2y^3}{-5x^3y}$.
19. $\frac{4a^2m^4x^5}{5a^5m^3x}$.
20. $\frac{42x^3y^2z^4}{7xy^2z^3}$.
21. $\frac{-3a^2b^3c^4d^5}{-a^4b^2cd^3}$.
22. $\frac{12am^5n^4p^3q^2}{4m^2n^3p^4q^5}$.

23. $(4a^2bz^3 \times 10a^2b^3z) \div 5a^3b^2z^2$.

24. $(21x^2y^4z^6 \div 3xy^2z)(-2x^3y^2z)$.

25. $104ab^3x^9 \div (91a^5b^6x^7 \div 7a^4b^4x)$.

26. $(24a^5b^3x \div 3a^2b^2) + (35a^6b^2x^2 \div -5a^3bx)$.

27. $85a^{4m+1} \div 5a^{4m-2}$. **28.** $84a^{n-4} \div 12a^2$.

60. To divide One Polynomial by Another.

If the divisor (one factor) $= a + b + c$,
and the quotient (other factor) $= n + p + q$,

then the dividend (product) $= \begin{cases} an + bn + cn \\ + ap + bp + cp \\ + aq + bq + cq. \end{cases}$

The first term of the dividend is an; that is, the product of a, the first term of the divisor, by n, the first term of the quotient. The first term n of the quotient is therefore found by dividing an, the first term of the dividend, by a, the first term of the divisor.

If the partial product formed by multiplying the entire divisor by n is subtracted from the dividend, the first term of the remainder ap is the product of a, the first term of the divisor, by p, the second term of the quotient; that is, the second term of the quotient is obtained by dividing the first term of the remainder by the first term of the divisor. In like manner, the third term of the quotient is obtained by dividing the first term of the new remainder by the first term of the divisor; and so on. Hence we have the following rule:

Arrange both the dividend and divisor in ascending or descending powers of some common letter.

Divide the first term of the dividend by the first term of the divisor.

Write the result as the first term of the quotient.

Multiply all the terms of the divisor by the first term of the quotient.

Subtract the product from the dividend.

If there be a remainder, consider it as a new dividend and proceed as before.

61. It is of fundamental importance to arrange the dividend and divisor *in the same order* with respect to a common letter, and *to keep this order throughout the operation.*

The beginner should study carefully the processes in the following examples:

(1) Divide $x^2 + 18x + 77$ by $x + 7$.

$$\begin{array}{l|l} x^2 + 18x + 77 & x + 7 \\ \cline{2-2} \underline{x^2 + 7x} & x + 11 \\ \phantom{x^2 +{}} 11x + 77 & \\ \phantom{x^2 +{}} \underline{11x + 77} & \end{array}$$

NOTE. The student will notice that by this process we have in effect separated the dividend into two parts, $x^2 + 7x$ and $11x + 77$, and divided each part by $x + 7$, and that the complete quotient is the sum of the partial quotients x and 11. Thus,

$$x^2 + 18x + 77 = x^2 + 7x + 11x + 77 = (x^2 + 7x) + (11x + 77);$$

$$\therefore \frac{x^2 + 18x + 77}{x + 7} = \frac{x^2 + 7x}{x + 7} + \frac{11x + 77}{x + 7} = x + 11.$$

(2) Divide $4a^4x^2 - 4a^2x^4 + x^6 - a^6$ by $x^2 - a^2$.

Arrange according to descending powers of x.

$$\begin{array}{l|l} x^6 - 4a^2x^4 + 4a^4x^2 - a^6 & x^2 - a^2 \\ \cline{2-2} \underline{x^6 - a^2x^4} & x^4 - 3a^2x^2 + a^4 \\ - 3a^2x^4 + 4a^4x^2 - a^6 & \\ \underline{- 3a^2x^4 + 3a^4x^2} & \\ \phantom{x^6 - 3a^2x^4 +{}} a^4x^2 - a^6 & \\ \phantom{x^6 - 3a^2x^4 +{}} \underline{a^4x^2 - a^6} & \end{array}$$

(3) Divide $22a^2b^2 + 15b^4 + 3a^4 - 10a^3b - 22ab^3$ by $a^2 + 3b^2 - 2ab$.

Arrange according to descending powers of a.

$$
\begin{array}{l|l}
3a^4 - 10a^3b + 22a^2b^2 - 22ab^3 + 15b^4 & a^2 - 2ab + 3b^2 \\ \cline{2-2}
\underline{3a^4 - 6a^3b + 9a^2b^2} & 3a^2 - 4ab + 5b^2 \\
\qquad - 4a^3b + 13a^2b^2 - 22ab^3 & \\
\qquad \underline{- 4a^3b + 8a^2b^2 - 12ab^3} & \\
\qquad\qquad\qquad 5a^2b^2 - 10ab^3 + 15b^4 & \\
\qquad\qquad\qquad \underline{5a^2b^2 - 10ab^3 + 15b^4} &
\end{array}
$$

(4) Divide $5x^3 - x + 1 - 3x^4$ by $1 + 3x^2 - 2x$.

Arrange according to ascending powers of x.

$$
\begin{array}{l|l}
1 - x + 5x^3 - 3x^4 & 1 - 2x + 3x^2 \\ \cline{2-2}
\underline{1 - 2x + 3x^2} & 1 + x - x^2 \\
\qquad x - 3x^2 + 5x^3 - 3x^4 & \\
\qquad \underline{x - 2x^2 + 3x^3} & \\
\qquad\quad - x^2 + 2x^3 - 3x^4 & \\
\qquad\quad \underline{ - x^2 + 2x^3 - 3x^4} &
\end{array}
$$

(5) Divide $x^3 + y^3 + z^3 - 3xyz$ by $x + y + z$.

Arrange according to descending powers of x.

$$
\begin{array}{l|l}
x^3 - 3yzx + y^3 + z^3 & x + y + z \\ \cline{2-2}
\underline{x^3 + yx^2 + zx^2} & x^2 - yx - zx + y^2 - yz + z^2 \\
\quad - yx^2 - zx^2 - 3yzx + y^3 + z^3 & \\
\quad \underline{- yx^2 - y^2x - yzx} & \\
\qquad\quad - zx^2 + y^2x - 2yzx + y^3 + z^3 & \\
\qquad\quad \underline{- zx^2 - yzx - z^2x} & \\
\qquad\qquad\qquad y^2x - yzx + z^2x + y^3 + z^3 & \\
\qquad\qquad\qquad \underline{y^2x + y^3 + y^2z} & \\
\qquad\qquad\qquad\quad - yzx + z^2x - y^2z + z^3 & \\
\qquad\qquad\qquad\quad \underline{- yzx - y^2z - yz^2} & \\
\qquad\qquad\qquad\qquad\qquad z^2x + yz^2 + z^3 & \\
\qquad\qquad\qquad\qquad\qquad \underline{z^2x + yz^2 + z^3} &
\end{array}
$$

Exercise 17.

Divide

1. $x^2 - 7x + 12$ by $x - 3$.
2. $x^2 + x - 72$ by $x + 9$.
3. $2x^3 - x^2 + 3x - 9$ by $2x - 3$.
4. $6x^3 + 14x^2 - 4x + 24$ by $2x + 6$.
5. $3x^2 + x + 9x^3 - 1$ by $3x - 1$.
6. $7x^3 + 58x - 24x^2 - 21$ by $7x - 3$.
7. $x^6 - 1$ by $x - 1$.
8. $a^3 - 2ab^2 + b^3$ by $a - b$.
9. $x^4 - 81y^4$ by $x - 3y$.
10. $x^5 - y^5$ by $x - y$.
11. $a^5 + 32b^5$ by $a + 2b$.
12. $2a^4 + 27ab^3 - 81b^4$ by $a + 3b$.
13. $x^4 + 11x^2 - 12x - 5x^3 + 6$ by $3 + x^2 - 3x$.
14. $x^4 - 9x^2 + x^3 - 16x - 4$ by $x^2 + 4 + 4x$.
15. $36 + x^4 - 13x^2$ by $6 + x^2 + 5x$.
16. $x^4 + 64$ by $x^2 + 4x + 8$.
17. $x^4 + x^3 + 57 - 35x - 24x^2$ by $x^2 - 3 + 2x$.
18. $1 - x - 3x^2 - x^5$ by $1 + 2x + x^2$.
19. $x^6 - 2x^3 + 1$ by $x^2 - 2x + 1$.
20. $a^4 + 2a^2b^2 + 9b^4$ by $a^2 - 2ab + 3b^2$.
21. $4x^5 - x^3 + 4x$ by $2 + 2x^2 + 3x$.
22. $a^5 - 243$ by $a - 3$.
23. $18x^4 + 82x^2 + 40 - 67x - 45x^3$ by $3x^2 + 5 - 4x$
24. $x^4 - 6xy - 9x^2 - y^2$ by $x^2 + y + 3x$.

25. $x^4 + 9x^2y^2 - 6x^3y - 4y^4$ by $x^2 - 3xy + 2y^2$.

26. $x^4 + x^2y^2 + y^4$ by $x^2 - xy + y^2$.

27. $x^5 + x^3 + x^4y + y^3 - 2xy^2 - x^3y^2$ by $x^3 + x - y$.

28. $2x^2 - 3y^2 + xy - xz - 4yz - z^2$ by $2x + 3y + z$.

29. $12 + 82x^2 + 106x^4 - 70x^5 - 112x^3 - 38x$
by $3 - 5x + 7x^2$.

30. $x^5 + y^5$ by $x^4 - x^3y + x^2y^2 - xy^3 + y^4$.

31. $2x^4 + 2x^2y^2 - 2xy^3 - 7x^3y - y^4$ by $2x^2 + y^2 - xy$.

32. $16x^4 + 4x^2y^2 + y^4$ by $4x^2 - 2xy + y^2$.

33. $32a^5b + 8a^3b^3 - ab^5 - 4a^2b^4 - 56a^4b^2$
by $b^3 - 4a^2b + 6ab^2$.

34. $1 + 5x^3 - 6x^4$ by $1 - x + 3x^2$.

35. $1 - 52a^4b^4 - 51a^3b^3$ by $4a^2b^2 + 3ab - 1$.

36. $x^7y - xy^7$ by $x^3y + 2xy^3 - 2x^2y^2 - y^4$.

37. $x^6 + 15x^4y^2 + 15x^2y^4 + y^6 - 6x^5y - 6xy^5 - 20x^3y^3$
by $x^3 - 3x^2y + 3xy^2 - y^3$.

38. $a^7 + 2a^3b^4 - 2a^4b^3 - 2a^6b - 6a^2b^5 - 3ab^6$
by $a^3 - 2a^2b - ab^2$.

39. $81x^6y + 18x^2y^5 - 54x^5y^2 - 18x^3y^4 - 18xy^6 - 9y^7$
by $3x^4 + x^2y^2 + y^4$.

40. $a^4 + 2a^3b + 8a^2b^2 + 8ab^3 + 16b^4$ by $a^2 + 4b^2$.

41. $8y^6 - x^6 + 21x^3y^3 - 24xy^5$ by $3xy - x^2 - y^2$.

42. $16a^4 + 9b^4 + 8a^2b^2$ by $4a^2 + 3b^2 - 4ab$.

43. $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.

44. $a^3 + 8b^3 + c^3 - 6abc$ by $a^2 + 4b^2 + c^2 - ac - 2ab - 2bc$.

45. $a^3 + b^3 + c^3 + 3a^2b + 3ab^2$ by $a + b + c$.

62. The operation of division may be shortened in some cases by the use of parentheses. Thus:

$$
\begin{array}{l|l}
x^3+(a+b+c)x^2+(ab+ac+bc)x+abc & x+b \\
\cline{2-2}
\underline{x^3+(\quad+b\quad)x^2\qquad\qquad\qquad\quad} & x^2+(a+c)x+ac \\
\qquad(a\quad+c)x^2+(ab+ac+bc)x & \\
\qquad\underline{(a\quad+c)x^2+(ab\qquad+bc)x\qquad\quad} & \\
\qquad\qquad\qquad\qquad\quad acx\qquad+abc & \\
\qquad\qquad\qquad\qquad\quad \underline{acx\qquad+abc} &
\end{array}
$$

Exercise 18.

Divide

1. $a^2(b+c)+b^2(a-c)+c^2(a-b)+abc$ by $a+b+c$.

2. $x^3-(a+b+c)x^2+(ab+ac+bc)x-abc$
   by $x^2-(a+b)x+ab$.

3. $x^3-2ax^2+(a^2+ab-b^2)x-a^2b+ab^2$ by $x-a+b$.

4. $x^4-(a^2-b-c)x^2-(b-c)ax+bc$ by x^2-ax+c.

5. $y^3-(m+n+p)y^2+(mn+mp+np)y-mnp$ by $y-p$.

6. $x^4+(5+a)x^3-(4-5a+b)x^2-(4a+5b)x+4b$
   by x^2+5x-4.

7. $x^4-(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2$
   $-(abc+abd+acd+bcd)x+abcd$
   by $x^2-(a+c)x+ac$.

8. $x^5-(m-c)x^4+(n-cm+d)x^3+(r+cn-dm)x^2$
   $+(cr+dn)x+dr$ by x^3-mx^2+nx+r.

9. $x^5-mx^4+nx^3-nx^2+mx-1$ by $x-1$.

10. $(x+y)^3+3(x+y)^2z+3(x+y)z^2+z^3$
    by $(x+y)^2+2(x+y)z+z^2$.

CHAPTER V.

SIMPLE EQUATIONS.

63. Equations. An equation is a statement in symbols that two expressions stand for the same number. Thus, the equation $3x + 2 = 8$ states that $3x + 2$ and 8 stand for the same number.

64. That part of the equation which precedes the sign of equality is called the **first member**, or **left side**, and that which follows the sign of equality is called the **second member**, or **right side.**

65. The statement of equality between two algebraic expressions, if true for all values of the letters involved, is called an **identical equation**; but if true only for certain particular values of the letters involved, it is called an **equation of condition.** Thus, $(a + b)^2 = a^2 + 2ab + b^2$, which is true for *all values* of a and b, is an *identical equation;* and $3x + 2 = 8$, which is true only when x stands for 2, is an *equation of condition.*

For brevity, an identical equation is called an **identity,** and an equation of condition is called simply an **equation.**

66. We often employ an equation to discover an *unknown number* from its relation to known numbers. We usually represent the unknown number by one of the *last* letters of the alphabet, as x, y, z; and, by way of distinction, we use the *first* letters, a, b, c, etc., to represent numbers that are supposed to be known, though not expressed in the number-

symbols of Arithmetic. Thus, in the equation $ax + b = c$, x is supposed to represent an unknown number, and a, b, and c are supposed to represent known numbers.

67. Simple Equations. An integral equation which contains the first power of the symbol for the unknown number, x, and no higher power, is called a **simple equation,** or an **equation of the first degree.** Thus, $ax + b = c$ is a simple equation, or an equation of the first degree *in* x.

68. Solution of an Equation. To solve an equation is to find the unknown number; that is, the number which, when substituted for its symbol in the given equation, renders the equation an identity. This number is said to *satisfy* the equation, and is called the **root** of the equation.

69. Axioms. In solving an equation, we make use of the following axioms:

Ax. 1. If equal numbers be added to equal numbers, the sums will be equal.

Ax. 2. If equal numbers be subtracted from equal numbers, the remainders will be equal.

Ax. 3. If equal numbers be multiplied by equal numbers, the products will be equal.

Ax. 4. If equal numbers be divided by equal numbers, the quotients will be equal.

If, therefore, the two sides of an equation be increased by, diminished by, multiplied by, or divided by equal numbers, the results will be equal.

Thus, if $8x = 24$, then $8x + 4 = 24 + 4$, $8x - 4 = 24 - 4$, $4 \times 8x = 4 \times 24$, and $8x \div 4 = 24 \div 4$.

70. Transposition of Terms. It becomes necessary in solving an equation to bring all the terms that contain the

symbol for the unknown number to one side of the equation, and all the other terms to the other side. This is called **transposing the terms.** We will illustrate by examples:

(1) Find the number for which x stands when

$$16x - 11 = 7x + 70.$$

First subtract $7x$ from both sides (Ax. 2), which gives

$$9x - 11 = 70.$$

Then add 11 to these equals (Ax. 1), which gives

$$9x = 81.$$

Divide both sides by 9 (Ax. 4),

$$x = 9.$$

(2) Find the number for which x stands when $x + b = a$.

The equation is $x + b = a$.

Subtract b from each side, $x + b - b = a - b$. (Ax. 2)

Since $+b$ and $-b$ in the left side cancel each other (§ 14), we have $x = a - b$.

(3) Find the number for which x stands when $x - b = a$.

The equation is $x - b = a$.

Add $+b$ to each side, $x - b + b = a + b$. (Ax. 1)

Since $-b$ and $+b$ in the left side cancel each other (§ 14), we have $x = a + b$.

71. The effect of the operation in the preceding equations, when Axioms (1) and (2) are used, is to take a term from one side and to put it on the other side with its sign changed. We can proceed in a like manner in any other case. Hence the general rule:

72. *Any term may be transposed from one side of an equation to the other provided its sign is changed.*

73. Any term, therefore, which occurs on both sides with the same sign may be removed from both without affecting the equality.

74. The sign of every term of an equation may be changed, for this is effected by multiplying by -1, which by Ax. 3 does not destroy the equality.

75. Verification. When the root is substituted for its symbol in the given equation, and the equation reduces to an *identity*, the root is said to be **verified.**

(1) What number added to twice itself gives 24?

Let x stand for the number;
then $2x$ will stand for twice the number,
and the number added to twice itself will be $x + 2x$.

But the number added to twice itself is 24;

$$\therefore x + 2x = 24.$$

Combining x and $2x$, $3x = 24$.

Divide by 3, the coefficient of x, $x = 8$. (Ax. 4)

VERIFICATION.

$$\begin{aligned} x + 2x &= 24, \\ 8 + 2 \times 8 &= 24, \\ 8 + 16 &= 24, \\ 24 &= 24. \end{aligned}$$

(2) If $4x - 5 = 19$, for what number does x stand?

We have the equation $4x - 5 = 19$.

Transpose -5 to the right side, $4x = 19 + 5$.

Combine, $4x = 24$.

Divide by 4, $x = 6$. (Ax. 4)

VERIFICATION.

$$\begin{aligned} 4x - 5 &= 19, \\ 4 \times 6 - 5 &= 19, \\ 24 - 5 &= 19, \\ 19 &= 19. \end{aligned}$$

(3) If $3x - 7$ stands for the same number as $14 - 4x$, what number does x stand for?

We have the equation

$$3x - 7 = 14 - 4x.$$

Transpose $-4x$ to the left side, and -7 to the right side,

$$3x + 4x = 14 + 7.$$

Combine, $\quad 7x = 21.$

Divide by 7, $\quad x = 3.$

VERIFICATION. $\quad 3x - 7 = 14 - 4x,$

$$3 \times 3 - 7 = 14 - 4 \times 3,$$

$$2 = 2.$$

(4) Solve the equation $(x - 3)(x - 4) = x(x - 1) - 30.$

We have $(x - 3)(x - 4) = x(x - 1) - 30.$

Remove the parentheses,

$$x^2 - 7x + 12 = x^2 - x - 30.$$

Since x^2 on the left and x^2 on the right are precisely the same, including the sign, they may be cancelled.

Then $\quad -7x + 12 = -x - 30.$

Transpose $-x$ to the left side, and $+12$ to the right side,

$$-7x + x = -30 - 12.$$

Combine, $\quad -6x = -42.$

Divide by -6, $\quad x = 7.$

VERIFICATION.

$$(7 - 3)(7 - 4) = 7(7 - 1) - 30,$$

$$4 \times 3 = 7 \times 6 - 30,$$

$$12 = 42 - 30,$$

$$12 = 12.$$

76. Hence, to solve an equation with one unknown number,

Transpose all the terms involving the unknown number to the left side, and all the other terms to the right side; combine the like terms, and divide both sides by the coefficient of the unknown number.

Exercise 19.

Find the value of x in

1. $5x - 1 = 19.$

2. $3x + 6 = 12.$

3. $24x = 7x + 34.$

4. $8x - 29 = 26 - 3x.$

5. $12 - 5x = 19 - 12x.$

6. $3x + 6 - 2x = 7x.$

7. $5x + 50 = 4x + 56.$

8. $16x - 11 = 7x + 70.$

9. $24x - 49 = 19x - 14.$

10. $3x + 23 = 78 - 2x.$

11. $26 - 8x = 80 - 14x.$

12. $13 - 3x = 5x - 3.$

13. $3x - 22 = 7x + 6.$

14. $8 + 4x = 12x - 16.$

15. $5x - (3x - 7) = 4x - (6x - 35).$

16. $6x - 2(9 - 4x) + 3(5x - 7) = 10x - (4 + 16x + 35).$

17. $9x - 3(5x - 6) + 30 = 0.$

18. $x - 7(4x - 11) = 14(x - 5) - 19(8 - x) - 61.$

19. $(x + 7)(x - 3) = (x - 5)(x - 15).$

20. $(x - 8)(x + 12) = (x + 1)(x - 6).$

21. $(x - 2)(7 - x) + (x - 5)(x + 3) - 2(x - 1) + 12 = 0.$

22. $(2x - 7)(x + 5) = (9 - 2x)(4 - x) + 229.$

23. $14 - x - 5(x - 3)(x + 2) + (5 - x)(4 - 5x) = 45x - 76.$

24. $(x + 5)^2 - (4 - x)^2 = 21x.$

25. $5(x - 2)^2 + 7(x - 3)^2 = (3x - 7)(4x - 19) + 42.$

77. Statement and Solution of Problems. The difficulties which the beginner usually meets in stating problems will be quickly overcome if he will observe the following directions:

Study the problem until you clearly understand its meaning and just what is required to be found.

Remember that x must not be put for money, length, time, weight, etc., but for the required *number* of *specified units* of money, length, time, weight, etc.

Express each statement carefully in algebraic language, and write out in words what each expression stands for.

Do not attempt to form the equation until all the statements are made in symbols.

We will illustrate by examples:

(1) John has three times as many oranges as James, and they together have 32. How many has each?

Let x be the *number* of oranges James has;
then $3x$ is the number of oranges John has;
and $x + 3x$ is the number of oranges they together have.

But 32 is the number of oranges they together have;

$$\therefore x + 3x = 32;$$

or, $$4x = 32,$$

and $$x = 8.$$

Since $x = 8$, $$3x = 24.$$

Therefore James has 8 oranges, and John has 24 oranges.

NOTE. Beginners in stating the preceding problem generally write:

Let $x =$ *what* James had.

Now, we know *what* James had. He had oranges, and we are to discover simply the *number* of oranges he had.

(2) James and John together have $24, and James has $8 more than John. How many dollars has each?

Let x be the number of dollars John has;
then $x + 8$ is the number of dollars James has;
and $x + (x + 8)$ is the number of dollars they both have.

But 24 is the number of dollars they both have;

$$\therefore x + (x + 8) = 24;$$

or,
$$x + x + 8 = 24.$$

Transpose and combine, $2x = 16$.

Divide by 2, $x = 8$.

Since $x = 8$, $x + 8 = 16$.

Therefore John has \$8, and James has \$16.

NOTE. The beginner must avoid the mistake of writing

Let x = John's money.

We are required to find the *number* of dollars John has, and therefore x must represent this required number.

(3) A and B had equal sums of money; B gave A \$5, and then 3 times A's money was equal to 11 times B's money. What had each at first?

Let x = number of dollars each had;
then $x + 5$ = number of dollars A had after receiving \$5;
and $x - 5$ = number of dollars B had after giving A \$5.

Since 3 times A's money is now equal to 11 times B's, we have therefore the equation:

$$3(x + 5) = 11(x - 5).$$

Removing parentheses, $3x + 15 = 11x - 55$.

Transposing, $3x - 11x = -55 - 15$.

Collecting, $-8x = -70$.

Dividing by -8, $x = 8\frac{3}{4}$.

Therefore, each had \$8.75.

(4) Find a number whose treble exceeds 50 by as much as its double falls short of 40.

Let $x =$ the number;
then $3x =$ its treble;
and $3x - 50 =$ the excess of its treble over 50;
also, $40 - 2x =$ the number its double lacks of 40.

Since the excess of $3x$ over 50 equals the number $2x$ lacks of 40, we have

$$3x - 50 = 40 - 2x;$$
$$3x + 2x = 40 + 50;$$
$$5x = 90;$$
$$x = 18.$$

Therefore the number required is 18.

(5) Find a number that exceeds 50 by 10 more than it falls short of 80.

Let $x =$ the required number;
then $x - 50 =$ its excess over 50;
and $80 - x =$ the number it lacks of 80.
Hence, $x - 50 - (80 - x) =$ the excess.
But $10 =$ the excess.

$$\therefore\ x - 50 - (80 - x) = 10,$$
or
$$x - 50 - 80 + x = 10.$$
$$\therefore\ 2x = 140,$$
and
$$x = 70.$$

Therefore the number required is 70.

Exercise 20.

1. To the double of a certain number I add 14, and obtain as a result 154. What is the number?

2. To four times a certain number I add 16, and obtain as a result 188. What is the number?

3. By adding 46 to a certain number, I obtain as a result a number three times as large as the original number. Find the original number.

4. One number is three times as large as another. If I take the smaller from 16 and the greater from 30, the remainders are equal. What are the numbers?

5. Divide the number 92 into four parts, such that the first exceeds the second by 10, the third by 18, and the fourth by 24.

6. The sum of two numbers is 20; and if three times the smaller number is added to five times the greater, the sum is 84. What are the numbers?

7. The joint ages of a father and son are 80 years. If the age of the son were doubled, he would be 10 years older than his father. What is the age of each?

8. A man has 6 sons, each 4 years older than the next younger. The eldest is three times as old as the youngest. What is the age of each?

9. Add $24 to a certain sum, and the amount will be as much above $80 as the sum is below $80. What is the sum?

10. Thirty yards of cloth and 40 yards of silk together cost $330; and the silk costs twice as much a yard as the cloth. How much does each cost a yard?

11. Find the number whose double increased by 24 exceeds 80 by as much as the number itself is less than 100.

12. The sum of $500 is divided among A, B, C, and D. A and B have together $280, A and C $260, and A and D $220. How much does each receive?

13. In a company of 266 persons composed of men, women, and children, there are twice as many men as women, and twice as many women as children. How many are there of each?

14. Find two numbers differing by 8, such that four times the less may exceed twice the greater by 10.

15. A is 58 years older than B, and A's age is as much above 60 as B's age is below 50. Find the age of each.

16. A man leaves his property, amounting to \$7500, to be divided among his wife, his two sons, and three daughters, as follows: a son is to have twice as much as a daughter, and the wife \$500 more than all the children together. How much will be the share of each?

17. A vessel containing some water was filled by pouring in 42 gallons, and there was then in the vessel seven times as much as at first. How much did the vessel hold?

18. A has \$72 and B has \$52. B gives A a certain sum; then A has three times as much as B. How much did A receive from B?

19. Divide 90 into two such parts that four times one part may be equal to five times the other.

20. Divide 60 into two such parts that one part exceeds the other by 24.

21. Divide 84 into two such parts that one part may be less than the other by 36.

NOTE. When we have to compare the ages of two persons at a given time, and also a number of years after or before the given time, we must remember that *both* persons will be so many years older or younger.

Thus, if x represent A's age, and $2x$ B's age, at the present time, A's age five years ago will be represented by $x-5$; and B's by $2x-5$. A's age five years hence will be represented by $x+5$; and B's age by $2x+5$.

22. A is twice as old as B, and 22 years ago he was three times as old as B. What is A's age?

23. A father is 30 and his son 6 years old. In how many years will the father be just twice as old as the son?

24. A is twice as old as B, and 20 years since he was three times as old. What is B's age?

25. A is three times as old as B, and 19 years hence he will be only twice as old as B. What is the age of each?

26. A man has three nephews; his age is 50, and the joint ages of the nephews is 42. How long will it be before the joint ages of the nephews will be equal to that of the uncle?

NOTE. In problems involving quantities of the same kind expressed in different units, we must be careful to reduce all the quantities to the *same unit*.

Thus, if x denote a number of inches, all the quantities of the same kind involved in the problem must be reduced to inches.

27. A sum of money consists of dollars and twenty-five-cent pieces, and amounts to \$20. The number of coins is 50. How many are there of each sort?

28. A person bought 30 pounds of sugar of two different kinds, and paid for the whole \$2.94. The better kind cost 10 cents a pound and the poorer kind 7 cents a pound. How many pounds were there of each kind?

29. A workman was hired for 40 days, at \$1 for every day he worked, but with the condition that for every day he did not work he was to pay 45 cents for his board. At the end of the time he received \$22.60. How many days did he work?

30. A wine merchant has two kinds of wine; one worth 50 cents a quart, and the other 75 cents a quart. From these he wishes to make a mixture of 100 gallons, worth \$2.40 a gallon. How many gallons must he take of each kind?

31. A gentleman gave some children 10 cents each, and had a dollar left. He found that he would have required one dollar more to enable him to give them 15 cents each. How many children were there?

32. Two casks contain equal quantities of vinegar; from the first cask 34 quarts are drawn, from the second 20 gallons; the quantity remaining in one vessel is now twice that in the other. How much did each cask contain at first?

33. A gentleman hired a man for 12 months, at the wages of \$90 and a suit of clothes. At the end of 7 months the man quits his service and receives \$33.75 and the suit of clothes. What was the price of the suit of clothes?

34. A man has three times as many quarters as half-dollars, four times as many dimes as quarters, and twice as many half-dimes as dimes. The whole sum is \$7.30. How many coins has he in all?

35. A person paid a bill of \$15.25 with quarters and half-dollars, and gave 51 pieces of money in all. How many of each kind were there?

36. A bill of 100 pounds was paid with guineas (21 shillings) and half-crowns ($2\frac{1}{2}$ shillings), and 48 more half-crowns than guineas were used. How many of each were paid?

CHAPTER VI.

MULTIPLICATION AND DIVISION.

Special Rules.

78. Special Rules of Multiplication. Some results of multiplication are of so great utility in shortening algebraic work that they should be carefully noticed and remembered. The following are important:

79. Square of the Sum of Two Numbers.

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a(a+b)+b(a+b)\\ &= a^2+ab+ab+b^2\\ &= a^2+2ab+b^2.\end{aligned}$$

Since a and b stand for *any* two numbers, we have

Rule 1. *The square of the sum of two numbers is the sum of their squares plus twice their product.*

80. Square of the Difference of Two Numbers.

$$\begin{aligned}(a-b)^2 &= (a-b)(a-b)\\ &= a(a-b)-b(a-b)\\ &= a^2-ab-ab+b^2\\ &= a^2-2ab+b^2.\end{aligned}$$

Hence we have

Rule 2. *The square of the difference of two numbers is the sum of their squares minus twice their product.*

81. Product of the Sum and Difference of Two Numbers.

$$\begin{aligned}(a+b)(a-b) &= a(a-b)+b(a-b)\\ &= a^2-ab+ab-b^2\\ &= a^2-b^2.\end{aligned}$$

Hence we have

RULE 3. *The product of the sum and difference of two numbers is the difference of their squares.*

82. Illustrations. If we put $2x$ for a and 3 for b, we have

Rule 1, $(2x+3)^2 = 4x^2+12x+9.$

Rule 2, $(2x-3)^2 = 4x^2-12x+9.$

Rule 3, $(2x+3)(2x-3) = 4x^2-9.$

If we are required to multiply $a+b+c$ by $a+b-c$, we may abridge the ordinary process as follows:

$$(a+b+c)(a+b-c) = [(a+b)+c][(a+b)-c]$$

By Rule 3, $= (a+b)^2-c^2$

By Rule 1, $= a^2+2ab+b^2-c^2.$

If we are required to multiply $a+b-c$ by $a-b+c$, we may put the expressions in the following forms, and perform the operation:

$$(a+b-c)(a-b+c) = [a+(b-c)][a-(b-c)]$$

By Rule 3, $= a^2-(b-c)^2$

By Rule 2, $= a^2-(b^2-2bc+c^2)$

$= a^2-b^2+2bc-c^2.$

Exercise 21.

Write the product of

1. $(x+y)^2$.
2. $(y-z)^2$.
3. $(2x+1)^2$.
4. $(2a+5b)^2$.
5. $(1-x^2)^2$.
6. $(3ax-4x^2)^2$.
7. $(1-7a)^2$.
8. $(5xy+2)^2$.
9. $(ab+cd)^2$.
10. $(3mn-4)^2$.
11. $(12+5x)^2$.
12. $(4xy^2-yz^2)^2$.
13. $(3abc-bcd)^2$.
14. $(4x^3-xy^2)^2$.
15. $(x+y)(x-y)$.
16. $(2a+b)(2a-b)$.
17. $1+a+b$ and $1-a-b$.
18. $1-a+b$ and $1+a-b$.
19. a^2+ab+b^2 and a^2-ab+b^2.
20. $3a+2b-c$ and $3a-2b+c$.

83. Square of any Polynomial. If we put x for a, and $y+z$ for b, in the identity

$$(a+b)^2 = a^2+2ab+b^2,$$

we shall have

$$[x+(y+z)]^2 = x^2+2x(y+z)+(y+z)^2,$$

or

$$\begin{aligned}(x+y+z)^2 &= x^2+2xy+2xz+y^2+2yz+z^2\\ &= x^2+y^2+z^2+2xy+2xz+2yz.\end{aligned}$$

It will be seen that the complete product consists of the sum of the squares of the terms of the given expression and twice the products of each term into all the terms that follow it.

Again, if we put $a-b$ for a, and $c-d$ for b, in the same identity,

$$(a+b)^2 = a^2 + 2ab + b^2,$$

we shall have

$$\begin{aligned}&[(a-b)+(c-d)]^2\\ &\quad = (a-b)^2 + 2(a-b)(c-d) + (c-d)^2\\ &\quad = (a^2-2ab+b^2) + 2a(c-d) - 2b(c-d) + (c^2-2cd+d^2)\\ &\quad = a^2 - 2ab + b^2 + 2ac - 2ad - 2bc + 2bd + c^2 - 2cd + d^2\\ &\quad = a^2 + b^2 + c^2 + d^2 - 2ab + 2ac - 2ad - 2bc + 2bd - 2cd.\end{aligned}$$

Here the same law holds as before, the sign of each double product being $+$ or $-$, according as the factors composing it have *like* or *unlike* signs. The same is true for any polynomial. Hence we have

RULE 4. *The square of a polynomial is the sum of the squares of the several terms and twice the products obtained by multiplying each term into all the terms that follow it.*

Exercise 22.

1. $(x+y+z)^2 =$
2. $(x-y+z)^2 =$
3. $(m+n-p-q)^2 =$
4. $(x^2+2x-3)^2 =$
5. $(x^2-6x+7)^2 =$
6. $(2x^2-7x+9)^2 =$
7. $(x^2+y^2-z^2)^2 =$
8. $(x^4-4x^2y^2+y^4)^2 =$
9. $(a^3+b^3+c^3)^2 =$
10. $(x^3-y^3-z^3)^2 =$
11. $(x+2y-3z)^2 =$
12. $(x^2-2y^2+5z^2)^2 =$
13. $(x^2+2x-2)^2 =$
14. $(x^2-5x+7)^2 =$
15. $(2x^2-3x-4)^2 =$
16. $(x+2y+3z)^2 =$

84. Product of Two Binomials. The product of two binomials which have the form $x+a$, $x+b$, should be carefully noticed and remembered.

(1)
$$\begin{aligned}(x+5)(x+3) &= x(x+3)+5(x+3)\\ &= x^2+3x+5x+15\\ &= x^2+8x+15.\end{aligned}$$

(2)
$$\begin{aligned}(x-5)(x-3) &= x(x-3)-5(x-3)\\ &= x^2-3x-5x+15\\ &= x^2-8x+15.\end{aligned}$$

(3)
$$\begin{aligned}(x+5)(x-3) &= x(x-3)+5(x-3)\\ &= x^2-3x+5x-15\\ &= x^2+2x-15.\end{aligned}$$

(4)
$$\begin{aligned}(x-5)(x+3) &= x(x+3)-5(x+3)\\ &= x^2+3x-5x-15\\ &= x^2-2x-15.\end{aligned}$$

Each of these results has three terms.

The first term of each result is the product of the first terms of the binomials.

The last term of each result is the product of the second terms of the binomials.

The middle term of each result has for a coefficient the *algebraic sum* of the second terms of the binomials.

The intermediate step given above may be omitted, and the products written at once by *inspection*. Thus,

(1) Multiply $x+8$ by $x+7$.

$$8+7=15,\quad 8\times 7=56.$$

$$\therefore\ (x+8)(x+7)=x^2+15x+56.$$

(2) Multiply $x-8$ by $x-7$.

$$(-8)+(-7)=-15,\quad (-8)(-7)=+56.$$

$$\therefore\ (x-8)(x-7)=x^2-15x+56.$$

(3) Multiply $x - 7y$ by $x + 6y$.

$$-7 + 6 = -1, \quad (-7y) \times 6y = -42y^2.$$

$$\therefore\ (x - 7y)(x + 6y) = x^2 - xy - 42y^2.$$

(4) Multiply $x^2 + 6(a + b)$ by $x^2 - 5(a + b)$.

$$+6 - 5 = 1, \quad 6(a + b) \times -5(a + b) = -30(a + b)^2.$$

$$\therefore\ [x^2+6(a+b)][x^2-5(a+b)] = x^4+(a+b)x^2-30(a+b)^2.$$

Exercise 23.

Write the product of

1. $(x + 2)(x + 3)$.
2. $(x + 1)(x + 5)$.
3. $(x - 3)(x - 6)$.
4. $(x - 8)(x - 1)$.
5. $(x - 8)(x + 1)$.
6. $(x - 2)(x + 5)$.
7. $(x - 3)(x + 7)$.
8. $(x - 2)(x - 4)$.
9. $(x + 1)(x + 11)$.
10. $(x - 2a)(x + 3a)$.
11. $(x - c)(x - d)$.
12. $(x - 4y)(x + y)$.
13. $(a - 2b)(a - 5b)$.
14. $(x^2 + 2y^2)(x^2 + y^2)$.
15. $(x^2 - 3xy)(x^2 + xy)$.
16. $(ax - 9)(ax + 6)$.
17. $(x + a)(x - b)$.
18. $(x - 11)(x + 4)$.
19. $(x + 12)(x - 11)$.
20. $(x - 10)(x - 5)$.

85. In like manner the product of *any* two binomials may be written. Thus,

(1) $$(2a - b)(3a + 4b) = 6a^2 + 8ab - 3ab - 4b^2$$
$$= 6a^2 + 5ab - 4b^2.$$

(2) $$(2x + 3y)(3x - 2y) = 6x^2 - 4xy + 9xy - 6y^2$$
$$= 6x^2 + 5xy - 6y^2.$$

86. Special Rules of Division. Some results in division are so important in abridging algebraic work that they should be carefully noticed and remembered.

87. Difference of Two Squares.

From § 81, $(a+b)(a-b)=a^2-b^2$.

$$\therefore\ \frac{a^2-b^2}{a+b}=a-b, \text{ and } \frac{a^2-b^2}{a-b}=a+b. \text{ Hence,}$$

RULE 1. *The difference of the squares of two numbers is divisible by the sum, and by the difference, of the numbers.*

Exercise 24.

Write the quotient of

1. $\dfrac{1-4x^2}{1-2x}$.
2. $\dfrac{1-4x^2}{1+2x}$.
3. $\dfrac{9a^2-b^2}{3a+b}$.
4. $\dfrac{9a^2-b^2}{3a-b}$.
5. $\dfrac{16a^2-9b^2}{4a+3b}$.
6. $\dfrac{1-9z^2}{1-3z}$.
7. $\dfrac{a^2b^2-c^2}{ab+c}$.
8. $\dfrac{4x^2-16y^2}{2x+4y}$.
9. $\dfrac{49-4a^2}{7-2a}$.
10. $\dfrac{x^2-81y^2}{x+9y}$.
11. $\dfrac{1-(x-y)^2}{1+(x-y)}$.
12. $\dfrac{a^2-(b+c)^2}{a+(b+c)}$.
13. $\dfrac{(x+y)^2-25}{(x+y)-5}$.
14. $\dfrac{1-(a-5b)^2}{1+(a-5b)}$.
15. $\dfrac{64-(2b+3c)^2}{8-(2b+3c)}$.
16. $\dfrac{(a-b)^2-(c-d)^2}{(a-b)+(c-d)}$.

88. Sum and Difference of Two Cubes. By performing the division, we find that

$$\frac{a^3+b^3}{a+b}=a^2-ab+b^2, \text{ and } \frac{a^3-b^3}{a-b}=a^2+ab+b^2.$$

Hence,

RULE 2. *The sum of the cubes of two numbers is divisible by the sum of the numbers, and the quotient is the sum of the squares of the numbers minus their product.*

RULE 3. *The difference of the cubes of two numbers is divisible by the difference of the numbers, and the quotient is the sum of the squares of the numbers plus their product.*

Exercise 25.

Write the quotient of

1. $(y^3-1)\div(y-1)$.
2. $(b^3-125)\div(b-5)$.
3. $(a^3-216)\div(a-6)$.
4. $(x^3-343)\div(x-7)$.
5. $(1-8x^3)\div(1-2x)$.
6. $(8a^3x^3-1)\div(2ax-1)$.
7. $(1-27x^3y^3)\div(1-3xy)$.
8. $(64a^3b^3-27x^3)\div(4ab-3x)$.
9. $(x^3+y^3)\div(x+y)$.
10. $(1+8a^3)\div(1+2a)$.
11. $(27a^3+b^3)\div(3a+b)$.
12. $(8a^3x^3+1)\div(2ax+1)$.
13. $(x^3+27y^3)\div(x+3y)$.
14. $(512x^3y^3+z^3)\div(8xy+z)$.
15. $(729a^3+216b^3)\div(9a+6b)$.
16. $(64a^3+1000b^3)\div(4a+10b)$.
17. $(64a^3b^3+27x^3)\div(4ab+3x)$.
18. $(x^3+343)\div(x+7)$.
19. $(27x^3y^3+8z^3)\div(3xy+2z)$.

89. Sum and Difference of any Two Like Powers. By performing the division, we find that

$$\frac{a^4 - b^4}{a - b} = a^3 + a^2b + ab^2 + b^3;$$

$$\frac{a^4 - b^4}{a + b} = a^3 - a^2b + ab^2 - b^3;$$

$$\frac{a^5 - b^5}{a - b} = a^4 + a^3b + a^2b^2 + ab^3 + b^4;$$

$$\frac{a^5 + b^5}{a + b} = a^4 - a^3b + a^2b^2 - ab^3 + b^4.$$

We find by trial that

$$a^2 + b^2,\ a^4 + b^4,\ a^6 + b^6, \text{ and so on,}$$

are *not* divisible by $a + b$ or by $a - b$. Hence,

If n is *any* positive integer,

(1) *$a^n + b^n$ is divisible by $a + b$ if n is odd, and by neither $a + b$ nor $a - b$ if n is even.*

(2) *$a^n - b^n$ is divisible by $a - b$ if n is odd, and by both $a + b$ and $a - b$ if n is even.*

NOTE. It is important to notice in the above examples that the terms of the quotient are all *positive* when the divisor is $a - b$, and *alternately positive and negative* when the divisor is $a + b$; also, that the quotient is homogeneous, the exponent of a decreasing and of b increasing by 1 for each successive term.

Exercise 26.

Find the quotient of

1. $(x^4 - y^4) \div (x - y)$.
2. $(x^4 - y^4) \div (x + y)$.
3. $(a^6 - x^6) \div (a - x)$.
4. $(a^6 - x^6) \div (a + x)$.
5. $(81a^4x^4 - 1) \div (3ax + 1)$.
6. $(64a^6 - b^6) \div (2a - b)$.
7. $(a^5 + 32b^5) \div (a + 2b)$.
8. $(a^5 - 32b^5) \div (a - 2b)$.
9. $(1 - 243a^5) \div (1 - 3a)$.
10. $(243a^5 + 1) \div (3a + 1)$.
11. $(x^7 - y^7) \div (x - y)$.
12. $(a^{10} - 1024) \div (a + 2)$.

CHAPTER VII.

FACTORS.

90. Rational Expressions. An expression is *rational* when none of its terms contain square or other roots.

91. Factors of Rational and Integral Expressions. By factors of a given integral number in Arithmetic we mean integral numbers that will divide the given number without remainder. Likewise by factors of a rational and integral expression in Algebra we mean rational and integral expressions that will divide the given expression without remainder.

92. Factors of Monomials. The factors of a monomial may be found by inspection. Thus, the factors of $14a^2b$ are 7, 2, a, a, and b.

93. Factors of Polynomials. The form of a polynomial that can be resolved into factors often suggests the process of finding the factors.

Case I.

94. When all the Terms have a Common Factor.

(1) Resolve into factors $2x^2 + 6xy$.

Since $2x$ is a factor of each term, we have

$$\frac{2x^2 + 6xy}{2x} = \frac{2x^2}{2x} + \frac{6xy}{2x} = x + 3y.$$

$$\therefore 2x^2 + 6xy = 2x(x + 3y).$$

Hence, the required factors are $2x$ and $x + 3y$.

(2) Resolve into factors $16a^3 + 4a^2 - 8a$.

Since $4a$ is a factor of each term, we have

$$\frac{16a^3 + 4a^2 - 8a}{4a} = \frac{16a^3}{4a} + \frac{4a^2}{4a} - \frac{8a}{4a}$$
$$= 4a^2 + a - 2.$$
$$\therefore\ 16a^3 + 4a^2 - 8a = 4a(4a^2 + a - 2).$$

Hence the required factors are $4a$ and $4a^2 + a - 2$.

Exercise 27.

Resolve into factors:

1. $5a^2 - 15a$.
2. $6a^3 + 18a^2 - 12a$.
3. $49x^2 - 21x + 14$.
4. $4x^3y - 12x^2y^2 + 8xy^3$.
5. $y^4 - ay^3 + by^2 + cy$.
6. $6a^5b^3 - 21a^4b^2 + 27a^3b^4$.
7. $54x^2y^6 + 108x^4y^8 - 243x^6y^9$.
8. $45x^7y^{10} - 90x^5y^7 - 360x^4y^8$.
9. $70a^3y^4 - 140a^2y^5 + 210ay^6$.
10. $32a^3b^6 + 96a^6b^8 - 128a^8b^9$.

Case II.

95. When the Terms can be grouped so as to show a Common Factor.

(1) Resolve into factors $ac + ad + bc + bd$.

$$ac + ad + bc + bd = (ac + ad) + (bc + bd) \qquad (1)$$
$$= a(c + d) + b(c + d) \qquad (2)$$
$$= (a + b)(c + d). \qquad (3)$$

Note. Since one factor is seen in (2) to be $c + d$, dividing by $c + d$ we obtain the other factor, $a + b$.

(2) Find the factors of $ac + ad - bc - bd$.

$$\begin{aligned} ac + ad - bc - bd &= (ac + ad) - (bc + bd) \\ &= a(c + d) - b(c + d) \\ &= (a - b)(c + d). \end{aligned}$$

NOTE. The signs of the last two terms, $-bc - bd$, are changed to $+$, when put within a parenthesis preceded by the minus sign.

(3) Resolve into factors $5x^3 - 15ax^2 - x + 3a$.

$$\begin{aligned} 5x^3 - 15ax^2 - x + 3a &= (5x^3 - 15ax^2) - (x - 3a) \\ &= 5x^2(x - 3a) - 1(x - 3a) \\ &= (5x^2 - 1)(x - 3a). \end{aligned}$$

(4) Resolve into factors $6y - 27x^2y - 10x + 45x^3$.

$$\begin{aligned} 6y - 27x^2y - 10x + 45x^3 &= (45x^3 - 27x^2y) - (10x - 6y) \\ &= 9x^2(5x - 3y) - 2(5x - 3y) \\ &= (9x^2 - 2)(5x - 3y). \end{aligned}$$

NOTE. By grouping the terms thus, $(6y - 27x^2y) - (10x - 45x^3)$, we obtain for the factors $(3y - 5x)(2 - 9x^2)$.

But $(3y - 5x)(2 - 9x^2) = (9x^2 - 2)(5x - 3y)$, since, by the Law of Signs, *the signs of two factors, or of any even number of factors, may be changed without altering the value of the product.*

Exercise 28.

Resolve into factors:

1. $x^2 - ax - bx + ab$.
2. $ab + ay - by - y^2$.
3. $bc + bx - cx - x^2$.
4. $mx + mn + ax + an$.
5. $cdx^2 - cxy + dxy - y^2$.
6. $abx - aby + pqx - pqy$.
7. $cdx^2 + adxy - bcxy - aby^2$.
8. $abcy - b^2dy - acdx + bd^2x$.
9. $ax - ay - bx + by$.
10. $cdz^2 - cyz + dyz - y^2$.

96. Perfect Squares. If an expression can be resolved into two equal factors, the expression is called a **perfect square,** and one of its equal factors is called its **square root.**

Thus, $16x^6y^2 = 4x^3y \times 4x^3y$. Hence, $16x^6y^2$ is a perfect square, and $4x^3y$ is its square root.

NOTE. The square root of $16x^6y^2$ may be $-4x^3y$ as well as $+4x^3y$, for $-4x^3y \times -4x^3y = 16x^6y^2$; but throughout this chapter the positive square root only will be considered.

97. The rule for extracting the square root of a perfect square, when the square is a monomial, is as follows:

Extract the square root of the coefficient, and divide the index of each letter by 2.

98. In like manner, the rule for extracting the cube root of a perfect cube, when the cube is a monomial, is,

Extract the cube root of the coefficient, and divide the index of each letter by 3.

99. By §§ 79, 80, a trinomial is a perfect square, if its first and last terms are perfect squares and positive, and its middle term is twice the product of their square roots. Thus, $16a^2 - 24ab + 9b^2$ is a perfect square.

The rule for extracting the square root of a perfect square, when the square is a trinomial, is as follows:

Extract the square roots of the first and last terms, and connect these square roots by the sign of the middle term.

Thus, if we wish to find the square root of

$$16a^2 - 24ab + 9b^2,$$

we take the square roots of $16a^2$ and $9b^2$, which are $4a$ and $3b$, respectively, and connect these square roots by the sign of the middle term, which is $-$. The square root is therefore

$$4a - 3b.$$

In like manner, the square root of

$$16a^2 + 24ab + 9b^2 \text{ is } 4a + 3b.$$

Case III.

100. When a Trinomial is a Perfect Square.

(1) Resolve into factors $x^2 + 2xy + y^2$.

From § 99, the factors of $x^2 + 2xy + y^2$ are

$$(x + y)(x + y).$$

(2) Resolve into factors $x^4 - 2x^2y + y^2$.

From § 99, the factors of $x^4 - 2x^2y + y^2$ are

$$(x^2 - y)(x^2 - y).$$

Exercise 29.

Resolve into factors:

1. $x^2 + 12x + 36$.
2. $x^2 + 28x + 196$.
3. $x^2 + 34x + 289$.
4. $z^2 + 2z + 1$.
5. $y^2 + 200y + 10{,}000$.
6. $z^4 + 14z^2 + 49$.
7. $x^2 + 36xy + 324y^2$.
8. $y^4 + 16y^2z^2 + 64z^4$.
9. $y^6 + 24y^3 + 144$.
10. $x^2z^2 + 162xz + 6561$.
11. $4a^2 + 12ab^2 + 9b^4$.
12. $9x^2y^4 + 30xy^2z + 25z^2$.
13. $9x^2 + 12xy + 4y^2$.
14. $4a^4x^2 + 20a^2x^3y + 25x^4y^2$.

Exercise 30.

Resolve into factors:

1. $a^2 - 8a + 16$.
2. $a^2 - 30a + 225$.
3. $x^2 - 38x + 361$.
4. $x^2 - 40x + 400$.
5. $y^2 - 100y + 2500$.
6. $y^4 - 20y^2 + 100$.
7. $y^2 - 50yz + 625z^2$.
8. $x^4 - 32x^2y^2 + 256y^4$.
9. $z^6 - 34z^3 + 289$.
10. $4x^4y^2 - 20x^2y^3z + 25y^4z^2$.
11. $16x^2y^4 - 8xy^3z^2 + y^2z^4$.
12. $9a^2b^2c^2 - 6ab^2c^2d + b^2c^2d^2$.

13. $16x^6 - 8x^4y^2 + x^2y^4$.

14. $a^6x^4 - 2a^3bx^2y^4 + b^2y^8$.

15. $36x^2y^2 - 60xy^3 + 25y^4$.

16. $1 - 6ab^3 + 9a^2b^6$.

17. $9m^2n^2 - 24mn + 16$.

18. $4b^2x^2 - 12bx^2y + 9x^2y^2$.

19. $49a^2 - 112ab + 64b^2$.

20. $64x^4y^6 - 160x^4y^3z + 100x^4z^2$.

21. $49a^2b^2c^2 - 28abcx + 4x^2$.

22. $121x^4 - 286x^2y + 169y^2$.

23. $289x^2y^2z^2 - 102xy^2z^2d + 9y^2z^2d^2$.

24. $361x^2y^2z^2 - 76abcxyz + 4a^2b^2c^2$.

Case IV.

101. When a Binomial is the Difference of Two Squares.

(1) Resolve into factors $x^2 - y^2$.

From § 81, $(x + y)(x - y) = x^2 - y^2$.

Hence, the difference of two squares is the product of two factors, which may be found as follows:

Take the square root of the first term and the square root of the second term.

The sum of these roots will form the first factor.

The difference of these roots will form the second factor.

102. If the squares are compound expressions, the same method may be employed.

(1) Resolve into factors $(x + 3y)^2 - 16a^2$.

The square root of the first term is $x + 3y$.
The square root of the second term is $4a$.
The sum of these roots is $x + 3y + 4a$.
The difference of these roots is $x + 3y - 4a$.
Therefore $(x + 3y)^2 - 16a^2 = (x + 3y + 4a)(x + 3y - 4a)$.

(2) Resolve into factors $a^2 - (3b - 5c)^2$.

The square roots of the terms are a and $(3b - 5c)$.
The sum of these roots is $a + (3b - 5c)$, or $a + 3b - 5c$.
The difference of these roots is $a - (3b - 5c)$, or $a - 3b + 5c$.
Therefore $a^2 - (3b - 5c)^2 = (a + 3b - 5c)(a - 3b + 5c)$.

103. If the factors contain like terms, these terms should be collected so as to present the results in the simplest form.

(3) Resolve into factors $(3a + 5b)^2 - (2a - 3b)^2$.

The square roots of the terms are $3a + 5b$ and $2a - 3b$.
The sum of these roots is $(3a + 5b) + (2a - 3b)$
or $3a + 5b + 2a - 3b = 5a + 2b$.
The difference of these roots is $(3a + 5b) - (2a - 3b)$,
or $3a + 5b - 2a + 3b = a + 8b$.
Therefore $(3a + 5b)^2 - (2a - 3b)^2 = (5a + 2b)(a + 8b)$.

104. By properly grouping the terms, compound expressions may often be written as the difference of two squares, and the factors readily found.

(1) Resolve into factors $a^2 - 2ab + b^2 - 9c^2$.

$$\begin{aligned} a^2 - 2ab + b^2 - 9c^2 &= (a^2 - 2ab + b^2) - 9c^2 \\ &= (a - b)^2 - 9c^2 \\ &= (a - b + 3c)(a - b - 3c). \end{aligned}$$

(2) Resolve into factors $12ab + 9x^2 - 4a^2 - 9b^2$.

Note. Here $12ab$ shows that it is the middle term of the expression which has in its first and last terms a^2 and b^2, and the minus sign before $4a^2$ and $9b^2$ shows that these terms must be put in a parenthesis with the minus sign before it, in order that they may be made positive.

The arrangement will be

$$\begin{aligned} 9x^2 - (4a^2 - 12ab + 9b^2) &= 9x^2 - (2a - 3b)^2 \\ &= (3x + 2a - 3b)(3x - 2a + 3b). \end{aligned}$$

(3) Resolve into factors $-a^2+b^2-c^2+d^2+2ac+2bd$.

NOTE. Here $2ac$, $2bd$, and $-a^2$, $-c^2$, indicate the arrangement required.

$$\begin{aligned}-a^2+b^2-c^2+d^2+2ac+2bd \\ &= (b^2+2bd+d^2)-(a^2-2ac+c^2) \\ &= (b+d)^2-(a-c)^2 \\ &= (b+d+a-c)(b+d-a+c).\end{aligned}$$

Exercise 31.

Resolve into factors:

1. a^2-b^2.
2. a^2-16.
3. $4a^2-25$.
4. a^4-b^4.
5. a^4-1.
6. a^8-b^8.
7. a^8-1.
8. $36x^2-49y^2$.
9. $100x^2y^2-121a^2b^2$.
10. $1-49x^2$.
11. a^4-25b^2.
12. $(a-b)^2-c^2$.
13. $x^2-(a-b)^2$.
14. $(a+b)^2-(c+d)^2$.
15. $(x+y)^2-(x-y)^2$.
16. $2ab-a^2-b^2+1$.
17. $x^2-2yz-y^2-z^2$.
18. $x^2-2xy+y^2-z^2$.
19. $a^2+12bc-4b^2-9c^2$.
20. $a^2-2ay+y^2-x^2-2xz-z^2$.
21. $2xy-x^2-y^2+z^2$.
22. $x^2+y^2-z^2-d^2-2xy-2dz$.
23. $x^2-y^2+z^2-a^2-2xz+2ay$.
24. $2ab+a^2+b^2-c^2$.
25. $2xy-x^2-y^2+a^2+b^2-2ab$.
26. $(ax+by)^2-1$.
27. $1-x^2-y^2+2xy$.
28. $(5a-2)^2-(a-4)^2$.
29. $a^2-2ab+b^2-x^2$.
30. $(x+1)^2-(y+1)^2$.
31. $(x+1)^2-(y-1)^2$.
32. $d^2-x^2+4xy-4y^2$.
33. $a^2-b^2-2bc-c^2$.
34. $4x^4-9x^2+6x-1$.

CASE V.

105. When a Trinomial has the Form $a^4+a^2b^2+b^4$.

A trinomial in the form of $a^4+a^2b^2+b^4$ can be written as the difference of two squares.

Since a trinomial is a perfect square when the middle term is *twice* the product of the square roots of the first and last terms, it is obvious that we must add a^2b^2 to the middle term of $a^4+a^2b^2+b^4$ to make it a perfect square. We must also subtract a^2b^2 to keep the value of the expression unchanged. We shall then have

$$\begin{aligned}(1)\quad a^4+a^2b^2+b^4 &= a^4+2a^2b^2+b^4-a^2b^2\\ &= (a^2+b^2)^2-a^2b^2\\ &= (a^2+b^2+ab)(a^2+b^2-ab)\\ &= (a^2+ab+b^2)(a^2-ab+b^2).\end{aligned}$$

If in the above expression we put 1 for b, we shall have

$$\begin{aligned}(2)\quad a^4+a^2+1 &= (a^4+2a^2+1)-a^2\\ &= (a^2+1)^2-a^2\\ &= (a^2+1+a)(a^2+1-a)\\ &= (a^2+a+1)(a^2-a+1).\end{aligned}$$

(3) Resolve into factors $4b^4-37b^2c^2+9c^4$.

Twice the product of the square roots of $4b^4$ and $9c^4$ is $12b^2c^2$. We may separate the term $-37b^2c^2$ into two terms, $-12b^2c^2$ and $-25b^2c^2$, and write the expression

$$\begin{aligned}&(4b^4-12b^2c^2+9c^4)-25b^2c^2\\ &\qquad= (2b^2-3c^2)^2-25b^2c^2\\ &\qquad= (2b^2-3c^2+5bc)(2b^2-3c^2-5bc)\\ &\qquad= (2b^2+5bc-3c^2)(2b^2-5bc-3c^2).\end{aligned}$$

Exercise 32.

Resolve into factors:

1. $x^4 + x^2y^2 + y^4$.
2. $9x^4 + 3x^2y^2 + 4y^4$.
3. $16x^4 - 17x^2y^2 + y^4$.
4. $81a^4 + 23a^2b^2 + 16b^4$.
5. $81a^4 - 28a^2b^2 + 16b^4$.
6. $9x^4 + 38x^2y^2 + 49y^4$.
7. $25a^4 - 9a^2b^2 + 16b^4$.
8. $49m^4 + 110m^2n^2 + 81n^4$.
9. $9a^4 + 21a^2c^2 + 25c^4$.
10. $49a^4 - 15a^2b^2 + 121b^4$.
11. $64x^4 + 128x^2y^2 + 81y^4$.
12. $4x^4 - 37x^2y^2 + 9y^4$.
13. $25x^4 - 41x^2y^2 + 16y^4$.
14. $81x^4 - 34x^2y^2 + y^4$.

Case VI.

106. When a Trinomial has the Form $x^2 + ax + b$.

From § 84 it is seen that a trinomial is often the product of two binomials. Conversely, a trinomial may, in certain cases, be resolved into two binomial factors.

107. If a trinomial of the form $x^2 + ax + b$ is such an expression that it can be resolved into two binomial factors, it is obvious that the first term of each factor will be x, and that the second terms of the factors will be two numbers whose product is b, the last term of the trinomial, and whose algebraic sum is a, the coefficient of x in the middle term of the trinomial.

(1) Resolve into factors $a^2 + 11a + 30$.

We are required to find two numbers whose product is 30 and whose sum is 11.

Two numbers whose product is 30 are 1 and 30, 2 and 15, 3 and 10, 5 and 6; and the sum of the last two numbers is 11. Hence,

$$a^2 + 11a + 30 = (a + 5)(a + 6).$$

(2) Resolve into factors $x^2 - 7x + 12$.

We are required to find two numbers whose product is 12 and whose algebraic sum is -7.

Since the product is $+12$, the two numbers are *both positive* or *both negative;* and since their sum is -7, they must both be negative.

Two negative numbers whose product is 12 are -12 and -1, -6 and -2, -4 and -3; and the sum of the last two numbers is -7. Hence,

$$x^2 - 7x + 12 = (x-4)(x-3).$$

(3) Resolve into factors $x^2 + 2x - 24$.

We are required to find two numbers whose product is -24 and whose algebraic sum is 2.

Since the product is -24, one of the numbers is positive and the other negative; and since their sum is $+2$, the larger number is positive.

Two numbers whose product is -24, and the larger number positive, are 24 and -1, 12 and -2, 8 and -3, 6 and -4; and the sum of the last two numbers is $+2$. Hence,

$$x^2 + 2x - 24 = (x+6)(x-4).$$

(4) Resolve into factors $x^2 - 3x - 18$.

We are required to find two numbers whose product is -18 and whose algebraic sum is -3.

Since the product is -18, one of the numbers is positive and the other negative; and since their sum is -3, the larger number is negative.

Two numbers whose product is -18, and the larger number negative, are -18 and 1, -9 and 2, -6 and 3; and the sum of the last two numbers is -3. Hence,

$$x^2 - 3x - 18 = (x-6)(x+3).$$

Therefore in general,

$$x^2 + (a+b)x + ab = (x+a)(x+b)$$

whatever the *values* of a and b.

Exercise 33.

Resolve into factors:

1. $x^2 + 11x + 24$.
2. $x^2 + 11x + 30$.
3. $y^2 + 17y + 60$.
4. $z^2 + 13z + 12$.
5. $x^2 + 21x + 110$.
6. $y^2 + 35y + 300$.
7. $b^2 + 23b + 102$.
8. $x^2 + 3x + 2$.
9. $x^2 + 7x + 6$.
10. $a^2 + 9ab + 8b^2$.
11. $x^2 + 13ax + 36a^2$.
12. $y^2 + 19py + 48p^2$.
13. $z^2 + 29qz + 100q^2$.
14. $a^4 + 5a^2 + 6$.
15. $z^6 + 4z^3 + 3$.
16. $a^2b^2 + 18ab + 32$.
17. $x^8y^4 + 7x^4y^2 + 12$.
18. $z^{10} + 10z^5 + 16$.
19. $a^2 + 9ab + 20b^2$.
20. $x^6 + 9x^3 + 20$.
21. $a^2x^2 + 14abx + 33b^2$.
22. $a^2c^2 + 7acx + 10x^2$.
23. $x^2y^2z^2 + 19xyz + 48$.
24. $b^2c^2 + 18abc + 65a^2$.
25. $r^2s^2 + 23rsz + 90z^2$.
26. $m^4n^4 + 20m^2n^2pq + 51p^2q^2$.

Exercise 34.

Resolve into factors:

1. $x^2 - 7x + 10$.
2. $x^2 - 29x + 190$.
3. $a^2 - 23a + 132$.
4. $b^2 - 30b + 200$.
5. $z^2 - 43z + 460$.
6. $x^2 - 7x + 6$.
7. $x^4 - 4a^2x^2 + 3a^4$.
8. $x^2 - 8x + 12$.
9. $z^2 - 57z + 56$.
10. $y^6 - 7y^3 + 12$.
11. $x^2y^2 - 27xy + 26$.
12. $a^4b^6 - 11a^2b^3 + 30$.

13. $a^2b^2c^2 - 13abc + 22.$

14. $x^2 - 15x + 50.$

15. $x^2 - 20x + 100.$

16. $a^2x^2 - 21ax + 54.$

17. $a^2x^2 - 16abx + 39b^2.$

18. $a^2c^2 - 24acz + 143z^2.$

19. $x^2 - 20x + 91.$

20. $x^2 - 23x + 120.$

21. $z^2 - 53z + 360.$

22. $x^2 - (a + c)x + ac.$

23. $y^2z^2 - 28abyz + 187a^2b^2.$

24. $c^2d^2 - 30abcd + 221a^2b^2.$

Exercise 35.

Resolve into factors:

1. $x^2 + 6x - 7.$

2. $x^2 + 5x - 84.$

3. $y^2 + 7y - 60.$

4. $y^2 + 12y - 45.$

5. $z^2 + 11z - 12.$

6. $z^2 + 13z - 140.$

7. $a^2 + 13a - 300.$

8. $a^2 + 25a - 150.$

9. $b^8 + 3b^4 - 4.$

10. $b^2c^2 + 3bc - 154.$

11. $c^{10} + 15c^5 - 100.$

12. $c^2 + 17c - 390.$

13. $a^2 + a - 132.$

14. $x^2y^2z^2 + 9xyz - 22.$

Exercise 36.

Resolve into factors:

1. $x^2 - 3x - 28.$

2. $y^2 - 7y - 18.$

3. $x^2 - 9x - 36.$

4. $z^2 - 11z - 60.$

5. $z^2 - 13z - 14.$

6. $a^2 - 15a - 100.$

7. $c^{10} - 9c^5 - 10.$

8. $x^2 - 8x - 20.$

9. $y^2 - 5ay - 50a^2.$

10. $a^2b^2 - 3ab - 4.$

11. $a^2x^2 - 3ax - 54.$

12. $c^2d^2 - 24cd - 180.$

13. $a^6c^2 - a^3c - 2.$

14. $y^8z^4 - 5y^4z^2 - 84.$

15. $a^2b^2 - 16ab - 36.$

16. $x^2 - (a-b)x - ab.$

Case VII.

108. When a Trinomial has the Form $ax^2 + bx + c$.

From § 85,

$$(3x-2)(5x+3)$$
$$= 15x^2 + 9x - 10x - 6 = 15x^2 - x - 6. \quad (1)$$
$$(3x-2)(5x-3)$$
$$= 15x^2 - 9x - 10x + 6 = 15x^2 - 19x + 6. \quad (2)$$

Consider the resulting trinomials:

The first term in (1) and (2) is the product $3x \times 5x$.

The middle term in (1) is the algebraic sum of the products

$3x \times 3$ and $(-2) \times 5x$.

The middle term in (2) is the algebraic sum of the products

$3x \times (-3)$ and $(-2) \times 5x$.

The last term in (1) is the product $(-2) \times 3$.

The last term in (2) is the product $(-2) \times (-3)$.

The trinomials have no monomial factor, since no one of their factors has a monomial factor. Hence,

1. If the *third* term of a given trinomial is *negative*, the *second* terms of its binomial factors will have *unlike signs*.

2. If the *third* term is *positive*, the *second* terms of its binomial factors will have the *same sign*, and this sign is the sign of the middle term.

3. If a trinomial has no monomial factor, neither of its binomial factors can have a monomial factor.

(1) Resolve into factors $6x^2 + 17x + 12$.

The first terms of the binomial factors must be either $6x$ and x, or $3x$ and $2x$.

The second terms of the binomial factors must be 12 and 1, or 6 and 2, or 3 and 4.

We therefore write

$$\text{I. } (6x + \quad)(x + \quad), \quad \text{or} \quad \text{II. } (3x + \quad)(2x + \quad).$$

For the second terms of these factors we must reject 1 and 12; for 12 put in the second factor of I. would make the product $6x \times 12$ too large, and put in the first factor of I., or in either factor of II., the result would show a monomial factor.

We must also reject 6 and 2; for if put in I. or II. the results would show monomial factors; and for the same reason we must reject 3 and 4 for I.

The required factors, therefore, are $(3x + 4)$ and $(2x + 3)$.

(2) Resolve into factors $14x^2 - 11x - 15$.

For a first trial we write

$$(7x \quad)(2x \quad).$$

Since the third term of the given trinomial is -15, the second terms of the binomial factors will have unlike signs, and the two products which together form the middle term will be one +, and the other −. Also, since the middle term is $-11x$, the negative product will exceed in absolute value the positive product by $-11x$.

The required factors, therefore, are $(7x + 5)$ and $(2x - 3)$.

Exercise 37.

Resolve into factors:

1. $12x^2 - 5x - 2$.
2. $12x^2 - 7x + 1$.
3. $12x^2 - x - 1$.
4. $3x^2 - 2x - 5$.
5. $3x^2 + 4x - 4$.
6. $6x^2 + 5x - 4$.
7. $4x^2 + 13x + 3$.
8. $4x^2 + 11x - 3$.
9. $4x^2 - 4x - 3$.
10. $x^2 - 3ax + 2a^2$.

11. $12a^4 + a^2x^2 - x^4$.

12. $2x^2 + 5xy + 2y^2$.

13. $6a^2x^2 + ax - 1$.

14. $6b^2 - 7bx - 3x^2$.

15. $4x^2 + 8x + 3$.

16. $a^2 - ax - 6x^2$.

17. $8a^2 + 14ab - 15b^2$.

18. $6a^2 - 19ac + 10c^2$.

19. $8x^2 + 34xy + 21y^2$.

20. $8x^2 - 22xy - 21y^2$.

21. $6x^2 + 19xy - 7y^2$.

22. $11a^2 - 23ab + 2b^2$.

23. $2c^2 - 13cd + 6d^2$.

24. $6y^2 + 7yz - 3z^2$.

Case VIII.

109. When a Binomial is the Sum or Difference of Two Cubes.

From § 88, $$\frac{a^3 + b^3}{a + b} = a^2 - ab + b^2;$$

and $$\frac{a^3 - b^3}{a - b} = a^2 + ab + b^2.$$

$$\therefore\ a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

and $$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

In like manner we can resolve into factors any expression which can be written as the sum or the difference of two cubes.

(1) Resolve into factors $8a^3 + 27b^6$.

Since by § 98, $8a^3 = (2a)^3$, and $27b^6 = (3b^2)^3$, we can write $8a^3 + 27b^6$ as $(2a)^3 + (3b^2)^3$.

Since $$a^3 + b^3 = (a + b)(a^2 - ab + b^2),$$

we have, by putting $2a$ for a and $3b^2$ for b,

$$(2a)^3 + (3b^2)^3 = (2a + 3b^2)(4a^2 - 6ab^2 + 9b^4).$$

(2) Resolve into factors $125x^3 - 1$.

$$\begin{aligned} 125x^3 - 1 &= (5x)^3 - 1 \\ &= (5x - 1)(25x^2 + 5x + 1). \end{aligned}$$

(3) Resolve into factors $x^6 + y^9$.

$$\begin{aligned} x^6 + y^9 &= (x^2)^3 + (y^3)^3 \\ &= (x^2 + y^3)(x^4 - x^2y^3 + y^6). \end{aligned}$$

110. The same method is applicable when the cubes are compound expressions.

(4) Resolve into factors $(x - y)^3 + z^3$.

Since $\quad a^3 + b^3 = (a + b)(a^2 - ab + b^2)$,

we have, by putting $x - y$ for a and z for b,

$$\begin{aligned} (x - y)^3 + z^3 &= [(x - y) + z][(x - y)^2 - (x - y)z + z^2] \\ &= (x - y + z)(x^2 - 2xy + y^2 - xz + yz + z^2). \end{aligned}$$

Exercise 38.

Resolve into factors :

1. $x^3 + 8$.
2. $x^3 + 216$.
3. $y^3 + 64z^3$.
4. $64b^3 + 125c^3$.
5. $8x^3 - 27y^3$.
6. $64y^3 - 1000z^3$.
7. $729x^3 - 512y^3$.
8. $27a^3 - 1728$.
9. $27a^3 - b^6$.
10. $(x + y)^3 - 1$.
11. $(x + y)^3 + 1$.
12. $8a^3 - (a - b)^3$.
13. $(x + y)^3 + c^3$.
14. $(x + y)^3 - (x - y)^3$.

Case IX.

111. When a Polynomial is the Product of Two Trinomials. The following method is convenient for resolving a polynomial into its trinomial factors:

Find the factors of

$$2x^2 - 5xy + 2y^2 + 7xz - 5yz + 3z^2.$$

1. Reject the terms that contain z.
2. Reject the terms that contain y.
3. Reject the terms that contain x.

Factor the expression that remains in each case.

1. $2x^2 - 5xy + 2y^2 = (x - 2y)(2x - y)$.
2. $2x^2 + 7xz + 3z^2 = (x + 3z)(2x + z)$.
3. $2y^2 - 5yz + 3z^2 = (2y - 3z)(y - z)$.

Arrange these three pairs of factors in two rows of three factors each, so that any two factors of each row may have a *common term including the sign.*

Thus, $\quad x - 2y,\ x + 3z,\ -2y + 3z;$
$\qquad\qquad 2x - y,\ 2x + z,\ -y + z.$

From the first row, select the *terms common to two factors* for one trinomial factor:

$$x - 2y + 3z.$$

From the second row, select the *terms common to two factors* for the other trinomial factor.

$$2x - y + z.$$

Hence, $2x^2 - 5xy + 2y^2 + 7xz - 5yz + 3z^2$
$$= (x - 2y + 3z)(2x - y + z).$$

112. When a factor obtained from the first three terms is also a factor of the remaining terms, the expression is easily resolved. Thus,

$$\begin{aligned} &x^2 - 3xy + 2y^2 - 3x + 6y \\ &= (x - 2y)(x - y) - 3(x - 2y) \\ &= (x - 2y)(x - y - 3). \end{aligned}$$

Exercise 39.

Resolve into factors:

1. $2x^2 - 5xy + 2y^2 - 17x + 13y + 21.$
2. $6x^2 - 37xy + 6y^2 - 5x - 5y - 1.$
3. $6x^2 - 5xy - 6y^2 - x - 5y - 1.$
4. $5x^2 - 8xy + 3y^2 + 7x - 5y + 2.$
5. $2x^2 - xy - 3y^2 - 8x + 7y + 6.$
6. $x^2 - 25y^2 - 10x - 20y + 21.$
7. $2x^2 - 5xy + 2y^2 - xz - yz - z^2.$
8. $6x^2 + xy - y^2 - 3xz + 6yz - 9z^2.$
9. $6x^2 - 7xy + y^2 + 35xz - 5yz - 6z^2.$
10. $5x^2 - 8xy + 3y^2 - 3xz + yz - 2z^2.$
11. $2x^2 - xy - 3y^2 - 5yz - 2z^2.$
12. $6x^2 - 13xy + 6y^2 + 12xz - 13yz + 6z^2.$
13. $x^2 - 2xy + y^2 + 5x - 5y.$
14. $2x^2 + 5xy - 3y^2 - 4xz + 2yz.$

Case X.

113. Binomials of the Form $x^n - y^n$ or $x^n + y^n$, and $n > 3$.

1. When a binomial has the form $x^n - y^n$, but cannot be written as the difference of two perfect squares, or of two perfect cubes, it is still possible to resolve it into two factors, one of which is $x - y$. Thus (§ 89),

$$a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4).$$

2. When a binomial has the form $x^n + y^n$, but cannot be written as the sum of two perfect cubes, it is still possible to resolve it into two factors, except when n is 2, 4, 8, 16, or some other power of 2. Thus (§ 89),

$$a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4).$$

But $a^2 + b^2$, $a^4 + b^4$, $a^8 + b^8$, cannot be resolved into *rational* factors.

NOTE 1. The student must be careful to select the best method of resolving an expression into factors. Thus, $a^6 - b^6$ can be written as the difference of two squares, or as the difference of two cubes, or be divided by $a - b$, or by $a + b$. Of all these methods, the best is to write the expression as the difference of two squares, as follows:

$$\begin{aligned}(a^3)^2 - (b^3)^2 &= (a^3 + b^3)(a^3 - b^3)\\ &= (a + b)(a^2 - ab + b^2)(a - b)(a^2 + ab + b^2).\end{aligned}$$

NOTE 2. From the last example, it will be seen that an expression can sometimes be resolved into three or more factors.

$$\begin{aligned}x^8 - b^8 &= (x^4 + b^4)(x^4 - b^4)\\ &= (x^4 + b^4)(x^2 + b^2)(x^2 - b^2)\\ &= (x^4 + b^4)(x^2 + b^2)(x + b)(x - b).\end{aligned}$$

NOTE 3. When a factor occurs in every term of an expression, this factor should first be removed. Thus,

$$\begin{aligned}8x^2 - 50a^2 + 4x - 10a &= 2(4x^2 - 25a^2 + 2x - 5a)\\ &= 2[(4x^2 - 25a^2) + (2x - 5a)]\\ &= 2(2x - 5a)(2x + 5a + 1).\end{aligned}$$

NOTE 4. Sometimes an expression can be easily resolved if we replace the last term but one by two terms, one of which shall have for a coefficient an exact divisor or a multiple of the last term. Thus,

(1) $$x^3 - 5x^2 + 11x - 15 = (x^3 - 5x^2 + 6x) + (5x - 15)$$
$$= x(x^2 - 5x + 6) + 5(x - 3)$$
$$= (x - 3)[x(x - 2) + 5]$$
$$= (x - 3)(x^2 - 2x + 5).$$

(2) $$x^3 - 26x - 5 = (x^3 - 25x) - (x + 5)$$
$$= x(x^2 - 25) - (x + 5)$$
$$= (x + 5)(x^2 - 5x - 1).$$

(3) $$x^3 + 3x^2 - 4 = (x^3 + 2x^2) + (x^2 - 4)$$
$$= x^2(x + 2) + (x^2 - 4)$$
$$= (x + 2)(x^2 + x - 2)$$
$$= (x + 2)(x + 2)(x - 1).$$

Exercise 40.

MISCELLANEOUS EXAMPLES.

Find the factors of

1. $5x^2 - 15x - 20.$
2. $2x^5 - 16x^4 + 24x^3.$
3. $3a^2b^2 - 9ab - 12.$
4. $a^2 + 2ax + x^2 + 4a + 4x.$
5. $a^2 - 2ab + b^2 - c^2.$
6. $x^2 - 2xy + y^2 - c^2 + 2cd - d^2.$
7. $4 - x^2 - 2x^3 - x^4.$
8. $a^2 - b^2 - a - b.$
9. $a^4 + a^2 + 1.$
10. $x^2 - y^2 - xz + yz.$
11. $ab - ac - b^2 + bc.$
12. $3x^2 - 3xz - xy + yz.$
13. $a^2 - x^2 - ab - bx.$
14. $a^2 - 2ax + x^2 + a - x.$
15. $3x^2 - 3y^2 - 2x + 2y.$
16. $x^4 + x^3 + x^2 + x.$
17. $a^4x^4 - a^3x^3 - a^2x^2 + 1.$
18. $3x^3 - 2x^2y - 27xy^2 + 18y^3.$

19. $4x^4 - x^2 + 2x - 1.$

20. $x^6 - y^6.$

21. $x^6 + y^6.$

22. $729 - x^6.$

23. $x^{12}y + y^{13}.$

24. $a^4c - c^5.$

25. $x^2 + 4x - 21.$

26. $3a^2 - 21ab + 30b^2.$

27. $2x^4 - 4x^3y - 6x^2y^2.$

28. $4a^2 - 4ab + b^2.$

29. $16x^2 - 80xy + 100y^2.$

30. $36a^2x^3y^2 - 25b^2x^5y^2.$

31. $9x^2y^4 - 30xy^2z + 25z^2.$

32. $16x^5 - x.$

33. $x^2 - 2xy - 2xz + y^2 + 2yz + z^2.$

34. $a^2 - ab - 6b^2 - 4a + 12b.$

35. $x^2 + 2xy + y^2 - x - y - 6.$

36. $(a+b)^4 - c^4.$

37. $x^2 - xy - 6y^2 - 4x + 12y.$

38. $1 - x + x^2 - x^3.$

39. $3x^2 - 11xy + 6y^2.$

40. $x^2 + 20x + 91.$

41. $(x-y)(x^2-z^2) - (x-z)(x^2-y^2).$

42. $x^2 - 5x - 24.$

43. $(x^2 - y^2 - z^2)^2 - 4y^2z^2.$

44. $5x^3y^2 + 5x^2yz - 60xz^2.$

45. $3x^3 - x^2 + 3x - 1.$

46. $x^2 - 2mx + m^2 - n^2.$

47. $4a^2b^2 - (a^2 + b^2 - c^2)^2.$

48. $a^7 + a^5.$

49. $1 - 14a^3x + 49a^6x^2.$

50. $y^2 - 4y - 117.$

51. $x^2 + 6x - 135.$

52. $4a^2 - 12ab + 9b^2 - 4c^2$

53. $(a + 3b)^2 - 9(b - c)^2.$

54. $9x^2 - 4y^2 + 4yz - z^2.$

55. $6b^2x^2 - 7bx^3 - 3x^4.$

56. $a^3 - b^3 - 3ab(a - b).$

57. $x^3 + y^3 + 3xy(x + y).$

58. $a^3 - b^3 - a(a^2 - b^2) + b(a - b)^2.$

59. $9x^2y^2 - 3xy^3 - 6y^4.$

60. $6x^2 + 13xy + 6y^2.$

61. $6a^2b^2 - ab^3 - 12b^4.$

62. $a^2 + 2ad + d^2 - 4b^2 + 12bc - 9c^2.$

63. $x^3 - 2x^2y + 4xy^2 - 8y^3.$

64. $4a^2x^2 - 8abx + 3b^2.$

65. $18x^2 - 24xy + 8y^2 + 9x - 6y.$

66. $2x^2 + 2xy - 12y^2 + 6xz + 18yz.$

67. $(x+y)^2 - 1 - xy(x+y+1).$

68. $x^2 - y^2 - z^2 + 2yz + x + y - z.$

69. $2x^2 + 4xy + 2y^2 + 2ax + 2ay.$

70. $16a^2b + 32abc + 12bc^2.$

71. $m^2p - m^2q - n^2p + n^2q.$

72. $12ax^2 - 14axy - 6ay^2.$

73. $2x^3 + 4x^2 - 70x.$

74. $16a^3x - 2x^4.$

75. $32bx^3 - 4by^3.$

76. $x - 27x^4.$

77. $x^{12} - y^{12}.$

78. $49m^2 - 121n^2.$

79. $16 - 81y^4.$

80. $12z^4 - z^2 - 6.$

81. $x^3 - x^2 + x - 1.$

82. $x^2 + 2x + 1 - y^2.$

83. $49(a-b)^2 - 64(m-n)^2.$

84. $4(ab+cd)^2 - (a^2 + b^2 - c^2 - d^2)^2.$

85. $x^2 - 53x + 360.$

86. $x^3 - 2x^2y + x^2 - 4x + 8y - 4.$

87. $2ab - 2bc - ae + ce + 2b^2 - be.$

88. $125x^5 + 350x^3y^2 + 245xy^4.$

89. $a^6 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + ab^5.$

90. $2a^4x - 2a^3cx + 2ac^3x - 2c^4x.$

91. $6x^2 - 5xy - 6y^2 + 3xz + 15yz - 9z^2.$

92. $4x^2 - 9xy + 2y^2 - 3xz - yz - z^2.$

93. $3a^2 - 7ab + 2b^2 + 5ac - 5bc + 2c^2.$

94. $x^4 - 2x^3 + x^2 - 8x + 8.$

95. $5x^2 - 8xy + 3y^2 - 5x + 3y.$

96. $a^2 - 2ad + d^2 - 4b^2 + 12bc - 9c^2.$

97. $(x^2 - x - 6)(x^2 - x - 20).$

CHAPTER VIII.

COMMON FACTORS AND MULTIPLES.

114. Common Factors. A **common factor** of two or more *numbers* is an integral number which divides each of them without a remainder.

115. A **common factor** of two or more *expressions* is an integral and rational expression which divides each of them without a remainder. Thus, $5a$ is a common factor of $20a$ and $25a$; $3x^2y^2$ is a common factor of $12x^2y^2$ and $15x^3y^3$.

116. Two *numbers* are said to be **prime** to each other when they have no common factor except 1.

117. Two *expressions* are said to be **prime** to each other when they have no common factor except 1.

118. The **highest common factor** of two or more *numbers* is the greatest number that will divide each of them without a remainder.

119. The **highest common factor** of two or more *expressions* is the expression of highest degree that will divide each of them without a remainder. Thus, $3a^2$ is the highest common factor of $3a^2$, $6a^3$, and $12a^4$; $5x^2y^2$ is the highest common factor of $10x^3y^2$ and $15x^2y^2$.

For brevity, we use H. C. F. to stand for "highest common factor."

To find the highest common factor of two algebraic expressions:

Case I.

120. When the Factors can be found by Inspection.

(1) Find the H. C. F. of $42a^3b^2$ and $60a^2b^4$.

$$42a^3b^2 = 2 \times 3 \times 7 \times aaa \times bb;$$

$$60a^2b^4 = 2 \times 2 \times 3 \times 5 \times aa \times bbbb.$$

$\therefore$ the H. C. F. $= 2 \times 3 \times aa \times bb$, or $6a^2b^2$.

(2) Find the H. C. F. of $2a^2x + 2ax^2$ and $3abxy + 3bx^2y$.

$$2a^2x + 2ax^2 = 2ax(a + x);$$

$$3abxy + 3bx^2y = 3bxy(a + x).$$

$\therefore$ the H. C. F. $= x(a + x)$.

(3) Find the H. C. F. of $4x^2 + 4x - 48$, $6x^2 - 48x + 90$.

$$\begin{aligned} 4x^2 + 4x - 48 &= 4(x^2 + x - 12) \\ &= 4(x - 3)(x + 4); \\ 6x^2 - 48x + 90 &= 6(x^2 - 8x + 15) \\ &= 6(x - 3)(x - 5). \end{aligned}$$

$\therefore$ the H. C. F. $= 2(x - 3)$

$= 2x - 6$.

Hence, to find the H. C. F. of two or more expressions:

Resolve each expression into its simplest factors.

Find the product of all the common factors, taking each factor the least number of times it occurs in any of the given expressions.

Exercise 41.

Find the H. C. F. of

1. $18ab^2c^2d$ and $36a^2bcd^2$.
2. $17pq^2$, $34p^2q$, and $51p^3q^3$.
3. $8x^2y^3z^4$, $12x^3y^2z^3$, and $20x^4y^3z^2$.
4. $30x^4y^5$, $90x^2y^3$, and $120x^3y^4$.
5. a^2-b^2 and a^3-b^3.
6. a^2-x^2 and $(a-x)^2$.
7. a^3+x^3 and $(a+x)^3$.
8. $9x^2-1$ and $(3x+1)^2$.
9. $7x^2-4x$ and $7a^2x-4a^2$.
10. $12a^3x^2y-4a^3xy^2$ and $30a^2x^3y^2-10a^2x^2y^3$.
11. $8a^3b^2c-12a^2bc^3$ and $6ab^4c+4ab^3c^2$.
12. x^2-2x-3 and x^2+x-12.
13. $2a^3-2ab^2$ and $4b(a+b)^2$.
14. $12x^3y(x-y)(x-3y)$ and $18x^2(x-y)(3x-y)$.
15. $3x^3+6x^2-24x$ and $6x^3-96x$.
16. $ac(a-b)(a-c)$ and $bc(b-a)(b-c)$.
17. $10x^3y-60x^2y^2+5xy^3$ and $5x^2y^2-5xy^3-100y^4$.
18. $x(x+1)^2$, $x^2(x^2-1)$, and $2x(x^2-x-2)$.
19. $3x^2-6x+3$, $6x^2+6x-12$, and $12x^2-12$.
20. $6(a-b)^4$, $8(a^2-b^2)^2$, and $10(a^4-b^4)$.
21. x^2-y^2, $(x+y)^2$, and $x^2+3xy+2y^2$.
22. x^2-y^2, x^3-y^3, and $x^2-7xy+6y^2$.
23. x^2-1, x^3-1, and x^2+x-2.

Case II.

121. When the Factors cannot be found by Inspection.

The method to be employed in this case is similar to that of the corresponding case in Arithmetic. And as in Arithmetic, pairs of continually decreasing numbers are obtained, which contain as a factor the H. C. F. required, so in Algebra, pairs of expressions of continually decreasing degrees are obtained, which contain as a factor the H. C. F. required.

122. The method depends upon the following principles:

(1) *Any factor of an expression is a factor also of any multiple of that expression.*

Thus, if c is contained 3 times in A, then c is contained 9 times in $3A$, and $3m$ times in mA.

(2) *Any common factor of two expressions is a factor of their sum, their difference, and of the sum or difference of any multiples of the expressions.*

Thus, if c is contained 5 times in A, and 3 times in B, then c is contained 8 times in $A+B$, and 2 times in $A-B$.

Also, in $5A+2B$ it is contained $5\times 5+2\times 3$, or 31 times, and in $5A-2B$ it is contained $5\times 5-2\times 3$, or 19 times.

(3) *The H. C. F. of two expressions is not changed if one of the expressions is divided by a factor that is not a factor of the other expression, or if one is multiplied by a factor that is not a factor of the other expression.*

Thus, the H. C. F. of $4a^2bc^2$ and a^2c^3d is not changed if we remove the factors 4 and b from $4a^2bc^2$, and d from a^2c^3d; or if we multiply $4a^2bc^2$ by 7, and a^2c^3d by 11.

123. We will first find the greatest common factor of twc arithmetical numbers, and then show that the same method is used in finding the H. C. F. of two algebraic expressions.

Find the greatest common factor of 18 and 48.

$$
\begin{array}{l}
18)\,48\,(2 \\
\underline{36} \\
12)\,18\,(1 \\
\underline{12} \\
6)\,12\,(2 \\
\underline{12}
\end{array}
$$

Since 6 is a factor of itself and of 12, it is, by (2), a factor of $6 + 12$, or 18.

Since 6 is a factor of 18, it is, by (1), a factor of 2×18, or 36; and, therefore, by (2), it is a factor of $36 + 12$, or 48.

Hence, 6 is a common factor of 18 and 48.

Again, every common factor of 18 and 48 is, by (1), a factor of 2×18, or 36; and, by (2), a factor of $48 - 36$, or 12.

Every such factor, being now a common factor of 18 and 12, is, by (2), a factor of $18 - 12$, or 6.

Therefore, the greatest common factor of 18 and 48 is contained in 6, and cannot be greater than 6. Hence 6, which has been shown to be a common factor of 18 and 48, is the **greatest** common factor of 18 and 48.

124. It will be seen that every remainder in the course of the operation contains the greatest common factor sought; and that this is the greatest factor common to that remainder and the preceding divisor. Hence,

The greatest common factor of any divisor and the corresponding dividend is the greatest common factor sought.

125. Let A and B stand for two algebraic expressions, arranged according to the descending powers of a common letter, the degree of B being not higher than that of A.

Let A be divided by B, and let Q stand for the quotient, and R for the remainder. Then

$$\begin{array}{l} B)\,A\,(Q \\ \quad\ \underline{BQ} \\ \quad\ \ R \end{array}$$

Whence, $R = A - BQ$, and $A = BQ + R$.

Any common factor of B and R will, by (2), be a factor of $BQ + R$, that is, of A; and any common factor of A and B will, by (2), be a factor of $A - BQ$, that is, of R.

Any common factor, therefore, of A and B is likewise a common factor of B and R. That is, the common factors of A and B are *the same* as the common factors of B and R; and therefore the H. C. F. of B and R is the H. C. F. of A and B.

If, now, we take the next step in the process, and divide B by R, and denote the remainder by S, then the H. C. F. of S and R can in a similar way be shown to be the same as the H. C. F. of B and R, and therefore the H. C. F. of A and B; and so on for each successive step. Hence,

The H. C. F. of any divisor and the corresponding dividend is the H. C. F. sought.

If at any step there is no remainder, the divisor is a factor of the corresponding dividend, and is therefore the H. C. F. of itself and the corresponding dividend. Hence, the *last divisor* is the H. C. F. sought.

NOTE. From the nature of division, the successive remainders are expressions of lower and lower degrees. Hence, unless at some step the division leaves no remainder, we shall at last have a remainder that does not contain the common letter. In this case the given expressions have no common factor.

Find the H. C. F. of $2x^2+x-3$ and $4x^3+8x^2-x-6$.

$$\begin{array}{l}
2x^2+x-3)\,4x^3+8x^2-\;x-6\,(2x+3\\
\qquad\qquad\quad \underline{4x^3+2x^2-6x}\\
\qquad\qquad\qquad\quad 6x^2+5x-6\\
\qquad\qquad\qquad\quad \underline{6x^2+3x-9}\\
\qquad\qquad\qquad\qquad\quad 2x+3)\,2x^2+\;x-3\,(x-1\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{2x^2+3x}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad -2x-3\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{-2x-3}
\end{array}$$

$\therefore$ the H. C. F. $=2x+3$.

Each division is continued until the first term of the remainder is of lower degree than that of the divisor.

126. This method is of use only to determine the **compound** factor of the H. C. F. **Simple** factors of the given expressions must first be separated from them, and the H. C. F. of these must be reserved to be multiplied into the compound factor obtained.

Find the H. C. F. of

$12x^4+30x^3-72x^2$ and $32x^3+84x^2-176x$.

$$12x^4+30x^3-72x^2=6x^2(2x^2+5x-12).$$

$$32x^3+84x^2-176x=4x(8x^2+21x-44).$$

$6x^2$ and $4x$ have $2x$ *common.*

$$\begin{array}{l}
2x^2+5x-12)\,8x^2+21x-44\,(4\\
\qquad\qquad\quad\;\; \underline{8x^2+20x-48}\\
\qquad\qquad\qquad\qquad x+\;4)\,2x^2+5x-12\,(2x-3\\
\qquad\qquad\qquad\qquad\qquad\quad \underline{2x^2+8x}\\
\qquad\qquad\qquad\qquad\qquad\qquad -3x-12\\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{-3x-12}
\end{array}$$

$\therefore$ the H. C. F. $=2x(x+4)$.

127. Modifications of this method are sometimes needed.

(1) Find the H. C. F. of $4x^2-8x-5$ and $12x^2-4x-65$.

$$\begin{array}{r} 4x^2-8x-5)\,12x^2-4x-65\,(3 \\ \underline{12x^2-24x-15} \\ 20x-50 \end{array}$$

The first division ends here, for $20x$ is of lower degree than $4x^2$. But if $20x-50$ is made the divisor, $4x^2$ will not contain $20x$ an *integral* number of times.

The H. C. F. sought *is contained in the remainder* $20x-50$, and is a *compound factor*. Hence if the *simple factor* 10 is removed, the H. C. F. must still be contained in $2x-5$, and therefore the process may be continued with $2x-5$ for a divisor.

$$\begin{array}{r} 2x-5)\,4x^2-8x-5\,(2x+1 \\ \underline{4x^2-10x} \\ 2x-5 \\ \underline{2x-5} \end{array}$$

$$\therefore \text{ the H. C. F.} = 2x-5.$$

(2) Find the H. C. F. of

$21x^3-4x^2-15x-2$ and $21x^3-32x^2-54x-7$.

$$\begin{array}{r} 21x^3-4x^2-15x-2)\,21x^3-32x^2-54x-7\,(1 \\ \underline{21x^3-4x^2-15x-2} \\ -28x^2-39x-5 \end{array}$$

The difficulty here cannot be obviated by *removing* a simple factor from the remainder, for $-28x^2-39x-5$ has no simple factor. In this case, the expression $21x^3-4x^2-15x-2$ must be *multiplied* by the simple factor 4 to make its first term exactly divisible by $-28x^2$.

The *introduction* of such a factor can in no way affect the H. C. F. sought, for 4 is not a factor of the remainder.

The *signs* of all the terms of the remainder may be changed; for if an expression A is divisible by $-F$, it is divisible by $+F$.

The process then is continued by changing the signs of the remainder and multiplying the divisor by 4.

$$
\begin{array}{lrl}
28x^2+39x+5) & 84x^3-16x^2-60x-8 & (3x \\
& \underline{84x^3+117x^2+15x\phantom{{}-8}} & \\
& -133x^2-75x-8 & \\
\text{Multiply by } -4, & \underline{-4} & \\
& 532x^2+300x+32 & (19 \\
& \underline{532x^2+741x+95} & \\
\text{Divide by } -63, & -63)\underline{-441x-63} & \\
& 7x+1 &
\end{array}
$$

$$
\begin{array}{rrl}
7x+1) & 28x^2+39x+5 & (4x+5 \\
& \underline{28x^2+4x\phantom{{}+5}} & \\
& 35x+5 & \\
\therefore \text{ the H. C. F.} = 7x+1. & \underline{35x+5} &
\end{array}
$$

(3) Find the H. C. F. of

$$8x^2+2x-3 \text{ and } 6x^3+5x^2-2.$$

$$
\begin{array}{lrl}
& 6x^3+5x^2-2 & \\
& \underline{4} & \\
8x^2+2x-3) & 24x^3+20x^2-8 & (3x+7 \\
& \underline{24x^3+6x^2-9x} & \\
& 14x^2+9x-8 & \\
\text{Multiply by } 4, & \underline{4} & \\
& 56x^2+36x-32 & \\
& \underline{56x^2+14x-21} & \\
\text{Divide by } 11, & 11)\underline{22x-11} &
\end{array}
$$

$$
\begin{array}{rrl}
2x-1) & 8x^2+2x-3 & (4x+3 \\
& \underline{8x^2-4x\phantom{{}-3}} & \\
& 6x-3 & \\
\therefore \text{ the H. C. F.} = 2x-1. & \underline{6x-3} &
\end{array}
$$

The following arrangement of the work will be found most convenient:

$$
\begin{array}{r|r|l}
8x^2+2x-3 & 6x^3+\ \ 5x^2-\ \ 2 \qquad\qquad & \\
\underline{8x^2-4x\qquad} & \underline{4 \qquad\qquad\qquad\qquad} & \\
6x-3 & 24x^3+20x^2-\ \ 8 \qquad\qquad & 3x \\
\underline{6x-3} & \underline{24x^3+\ \ 6x^2-\ \ 9x} \qquad & \\
 & 14x^2+\ \ 9x-\ \ 8 & \\
 & \underline{4 \qquad\qquad\qquad\quad} & \\
 & 56x^2+36x-32 & +7 \\
 & \underline{56x^2+14x-21} & \\
 & 11)\underline{22x-11} & \\
 & 2x-\ \ 1 & 4x+3
\end{array}
$$

128. From the foregoing examples it will be seen that, in the algebraic process of finding the H. C. F., the following steps, in the order here given, must be carefully observed:

I. Simple factors of the given expressions are to be removed from them, and the H. C. F. of these is to be reserved as a factor of the H. C. F. sought.

II. The resulting compound expressions are to be arranged according to the *descending* powers of a common letter; and that expression which is of the lower degree is to be taken for the divisor; or, if both are of the same degree, that whose first term has the smaller coefficient.

III. Each division is to be continued until the remainder is of lower degree than the divisor.

IV. If the final remainder of any division is found to contain a factor that is not a *common* factor of the given expressions, *this factor is to be removed;* and the resulting expression is to be used as the next divisor.

V. A dividend whose first term is not exactly divisible by the first term of the divisor, is to be *multiplied* by such a number as will make it thus divisible.

Exercise 42.

Find the H. C. F. of

1. $5x^2 + 4x - 1,\ 20x^2 + 21x - 5.$
2. $2x^3 - 4x^2 - 13x - 7,\ 6x^3 - 11x^2 - 37x - 20.$
3. $6a^4 + 25a^3 - 21a^2 + 4a,\ 24a^4 + 112a^3 - 94a^2 + 18a.$
4. $9x^3 + 9x^2 - 4x - 4,\ 45x^3 + 54x^2 - 20x - 24.$
5. $27x^6 - 3x^4 + 6x^3 - 3x^2,\ 162x^6 + 48x^3 - 18x^2 + 6x.$
6. $20x^3 - 60x^2 + 50x - 20,\ 32x^4 - 92x^3 + 68x^2 - 24x.$
7. $4x^2 - 8x - 5,\ 12x^2 - 4x - 65.$
8. $3a^3 - 5a^2x - 2ax^2,\ 9a^3 - 8a^2x - 20ax^2.$
9. $10x^3 + x^2 - 9x + 24,\ 20x^4 - 17x^2 + 48x - 3.$
10. $8x^3 - 4x^2 - 32x - 182,\ 36x^3 - 84x^2 - 111x - 126.$
11. $5x^2(12x^3 + 4x^2 + 17x - 3),\ 10x(24x^3 - 52x^2 + 14x - 1).$
12. $9x^4y - x^2y^3 - 20xy^4,\ 18x^3y - 18x^2y^2 - 2xy^3 - 8y^4.$
13. $6x^2 - x - 15,\ 9x^2 - 3x - 20.$
14. $12x^3 - 9x^2 + 5x + 2,\ 24x^2 + 10x + 1.$
15. $6x^3 + 15x^2 - 6x + 9,\ 9x^3 + 6x^2 - 51x + 36.$
16. $4x^3 - x^2y - xy^2 - 5y^3,\ 7x^3 + 4x^2y + 4xy^2 - 3y^3.$
17. $2a^3 - 2a^2 - 3a - 2,\ 3a^3 - a^2 - 2a - 16.$
18. $12y^3 + 2y^2 - 94y - 60,\ 48y^3 - 24y^2 - 348y + 30.$
19. $9x(2x^4 - 6x^3 - x^2 + 15x - 10),$
    $6x^2(4x^4 + 6x^3 - 4x^2 - 15x - 15).$
20. $15x^4 + 2x^3 - 75x^2 + 5x + 2,\ 35x^4 + x^3 - 175x^2 + 30x + 1.$
21. $21x^4 - 4x^3 - 15x^2 - 2x,\ 21x^3 - 32x^2 - 54x - 7.$
22. $9x^4y - 22x^2y^3 - 3xy^4 + 10y^5,\ 9x^5y - 6x^4y^2 + x^3y^3 - 25xy^5.$

23. $6x^5 - 4x^4 - 11x^3 - 3x^2 - 3x - 1,$
$4x^4 + 2x^3 - 18x^2 + 3x - 5.$

24. $x^4 - ax^3 - a^2x^2 - a^3x - 2a^4,\ 3x^3 - 7ax^2 + 3a^2x - 2a^3.$

129. The H. C. F. of three expressions may be obtained by resolving them into their prime factors; or by finding the H. C. F. of two of them, and then of that and the third expression.

For, if A, B, and C are three expressions,
and D the highest common factor of A and B,
and E the highest common factor of D and C,
Then D contains every factor common to A and B,
and E contains every factor common to D and C.
$\therefore$ E contains every factor common to A, B, and C.

Exercise 43.

Find the H. C. F. of

1. $2x^2 + x - 1,\ x^2 + 5x + 4,\ x^3 + 1.$

2. $y^3 - y^2 - y + 1,\ 3y^2 - 2y - 1,\ y^3 - y^2 + y - 1.$

3. $x^3 - 4x^2 + 9x - 10,\ x^3 + 2x^2 - 3x + 20,\ x^3 + 5x^2 - 9x + 35.$

4. $x^3 - 7x^2 + 16x - 12,\ 3x^3 - 14x^2 + 16x,$
$5x^3 - 10x^2 + 7x - 14.$

5. $y^3 - 5y^2 + 11y - 15,\ y^3 - y^2 + 3y + 5,$
$2y^3 - 7y^2 + 16y - 15.$

6. $2x^2 + 3x - 5,\ 3x^2 - x - 2,\ 2x^2 + x - 3.$

7. $x^3 - 1,\ x^3 - x^2 - x - 2,\ 2x^3 - x^2 - x - 3.$

8. $x^3 - 3x - 2,\ 2x^3 + 3x^2 - 1,\ x^3 + 1.$

9. $12(x^4 - y^4),\ 10(x^6 - y^6),\ 8(x^4y + xy^4).$

10. $x^4 + xy^3,\ x^3y + y^4,\ x^4 + x^2y^2 + y^4.$

11. $2(x^2y - xy^2),\ 3(x^3y - xy^3),\ 4(x^4y - xy^4),\ 5(x^5y - xy^5).$

130. Common Multiples. A **common multiple** of two or more *expressions* is an expression which is exactly divisible by each of the expressions.

The **lowest common multiple** of two or more *expressions* is the expression of lowest degree that is exactly divisible by each of the given expressions. Thus, $24(x^2-y^2)$ is the lowest common multiple of $3(x-y)$ and $8(x+y)$.

We use L. C. M. to stand for "lowest common multiple."

To find the L. C. M. of two or more algebraic expressions:

Case I.

131. When the Factors of the Expressions can be found by Inspection.

(1) Find the L. C. M. of $42a^3b^2$ and $60a^2b^4$.

$$42a^3b^2 = 2\times 3\times 7\times a^3\times b^2;$$
$$60a^2b^4 = 2\times 2\times 3\times 5\times a^2\times b^4.$$

The L. C. M. must evidently contain each factor the greatest number of times that it occurs in either expression.

$$\begin{aligned}\therefore\ \text{L. C. M.} &= 2\times 2\times 3\times 7\times 5\times a^3\times b^4,\\ &= 420a^3b^4.\end{aligned}$$

(2) Find the L. C. M. of

$$4x^2+4x-48,\ 6x^2-48x+90,\ 4x^2-10x-6.$$

$$\begin{aligned}4x^2+4x-48 &= 4(x^2+x-12) = 2\times 2(x-3)(x+4);\\ 6x^2-48x+90 &= 6(x^2-8x+15) = 2\times 3(x-3)(x-5);\\ 4x^2-10x-6 &= 2(2x^2-5x-3) = 2(x-3)(2x+1).\end{aligned}$$

$$\therefore\ \text{L. C. M.} = 2\times 2\times 3\times(x-3)(x+4)(x-5)(2x+1).$$

Hence, to find the L. C. M. of two or more expressions:

Resolve each expression into its simplest factors.

Find the product of all the different factors, taking each factor the greatest number of times it occurs in any of the given expressions.

Exercise 44.

Find the L. C. M. of

1. $4a^3x,\ 6a^2x^2,\ 2ax^2$.
2. $18ax^2,\ 72ay^2,\ 12xy$.
3. $x^2,\ ax+x^2$.
4. $x^2-1,\ x^2-x$.
5. $a^2-b^2,\ a^2+ab$.
6. $2x-1,\ 4x^2-1$.
7. $a+b,\ a^3+b^3$.
8. $x^2-1,\ x^2+1,\ x^4-1$.
9. $x^2-x,\ x^3-1,\ x^3+1$.
10. $x^2-1,\ x^2-x,\ x^3-1$.
11. $2a+1,\ 4a^2-1,\ 8a^3+1$.
12. $(a+b)^2,\ a^2-b^2$.
13. $4(1+x),\ 4(1-x),\ 2(1-x^2)$.
14. $x-1,\ x^2+x+1,\ x^3-1$.
15. $x^2-y^2,\ (x+y)^2,\ (x-y)^2$.
16. $x^2-y^2,\ 3(x-y)^2,\ 12(x^3+y^3)$.
17. $6(x^2+xy),\ 8(xy-y^2),\ 10(x^2-y^2)$.
18. $x^2+5x+6,\ x^2+6x+8$.
19. $a^2-a-20,\ a^2+a-12$.
20. $x^2+11x+30,\ x^2+12x+35$.
21. $x^2-9x-22,\ x^2-13x+22$.
22. $4ab(a^2-3ab+2b^2),\ 5a^2(a^2+ab-6b^2)$.
23. $20(x^2-1),\ 24(x^2-x-2),\ 16(x^2+x-2)$.
24. $12xy(x^2-y^2),\ 2x^2(x+y)^2,\ 3y^2(x-y)^2$.
25. $(a-b)(b-c),\ (b-c)(c-a),\ (c-a)(a-b)$.
26. $(a-b)(a-c),\ (b-a)(b-c),\ (c-a)(c-b)$.
27. $x^3-4x^2+3x,\ x^4+x^3-12x^2,\ x^5+3x^4-4x^3$.
28. $x^2y-xy^2,\ 3x(x-y)^2,\ 4y(x-y)^3$.
29. $(a+b)^2-(c+d)^2,\ (a+c)^2-(b+d)^2,\ (a+d)^2-(b+c)^2$.
30. $(2x-4)(3x-6),\ (x-3)(4x-8),\ (2x-6)(5x-10)$.

Case II.

132. When the Factors of the Expressions cannot be found by Inspection.

In this case the factors of the given expressions may be found by finding their H. C. F. and dividing each expression by this H. C. F.

Find the L. C. M. of

$$6x^3 - 11x^2y + 2y^3 \text{ and } 9x^3 - 22xy^2 - 8y^3.$$

$$\begin{array}{r|r|l}
6x^3 - 11x^2y \qquad\qquad + 2y^3 & 9x^3 - 22xy^2 - \ \ 8y^3 & 3 \\
\underline{6x^3 - \ \ 8x^2y - 4xy^2 \qquad\quad} & \underline{2 \qquad\qquad\qquad\qquad\quad} & \\
-\ \ 3x^2y + 4xy^2 + 2y^3 & 18x^3 - 44xy^2 - 16y^3 & \\
\underline{-\ \ 3x^2y + 4xy^2 + 2y^3} & \underline{18x^3 - 33x^2y + \ \ 6y^3} & \\
 & 11y\underline{)33x^2y - 44xy^2 - 22y^3} & \\
 & 3x^2 \ \ - \ \ 4xy \ \ - \ \ 2y^2 & 2x - y
\end{array}$$

$$\therefore \text{ the H. C. F.} = 3x^2 - 4xy - 2y^2.$$

Hence, $\qquad 6x^3 - 11x^2y + 2y^3 = (2x - y)(3x^2 - 4xy - 2y^2),$

and $\qquad 9x^3 - 22xy^2 - 8y^3 = (3x + 4y)(3x^2 - 4xy - 2y^2).$

$$\therefore \text{ the L. C. M.} = (2x - y)(3x + 4y)(3x^2 - 4xy - 2y^2).$$

133. The product of the H. C. F. and the L. C. M. of two expressions is equal to the product of the given expressions.

For, let A and B denote the two expressions, and D their H. C. F. Suppose $A = aD$, and $B = bD$.

Since D consists of all the factors common to A and B, a and b have no common factor, and L. C. M. of a and b is ab.

Hence, the L. C. M. of aD and bD is abD.

Now, $\qquad A = aD$, and $B = bD$.

$$\therefore AB = abD^2.$$

$$\therefore \frac{AB}{D} = abD = \text{the lowest common multiple.}$$

Hence, *the L. C. M. of two expressions can be found by dividing their product by their H. C. F.*

134. To find the L. C. M. of *three* expressions A, B, C. Find M, the L. C. M. of A and B; then the L. C. M. of M, and C is the L. C. M. required.

Exercise 45.

Find the L. C. M. of

1. $6x^2 - x - 2,\ 21x^2 - 17x + 2,\ 14x^2 + 5x - 1.$
2. $x^2 - 1,\ x^2 + 2x - 3,\ 6x^2 - x - 2.$
3. $x^3 - 27,\ x^2 - 15x + 36,\ x^3 - 3x^2 - 2x + 6.$
4. $5x^2 + 19x - 4,\ 10x^2 + 13x - 3.$
5. $12x^2 + xy - 6y^2,\ 18x^2 + 18xy - 20y^2.$
6. $x^4 - 2x^3 + x,\ 2x^4 - 2x^3 - 2x - 2.$
7. $12x^2 + 2x - 4,\ 12x^2 - 42x - 24,\ 12x^2 - 28x - 24.$
8. $x^3 - 6x^2 + 11x - 6,\ x^3 - 9x^2 + 26x - 24,$
   $x^3 - 8x^2 + 19x - 12.$
9. $x^2 - 4a^2,\ x^3 + 2ax^2 + 4a^2x + 8a^3,\ x^3 - 2ax^2 + 4a^2x - 8a^3.$
10. $x^3 + 2x^2y - xy^2 - 2y^3,\ x^3 - 2x^2y - xy^2 + 2y^3.$
11. $1 + p + p^2,\ 1 - p + p^2,\ 1 + p^2 + p^4.$
12. $(1 - a),\ (1 - a)^2,\ (1 - a)^3.$
13. $(a + c)^2 - b^2,\ (a + b)^2 - c^2,\ (b + c)^2 - a^2.$
14. $3c^3 - 3c^2y + cy^2 - y^3,\ 4c^3 - c^2y - 3cy^2.$
15. $m^3 - 8m + 3,\ m^6 + 3m^5 + m + 3.$
16. $20n^4 + n^2 - 1,\ 25n^4 + 5n^3 - n - 1.$
17. $b^4 - 2b^3 + b^2 - 8b + 8,\ 4b^3 - 12b^2 + 9b - 1.$
18. $2r^5 - 8r^4 + 12r^3 - 8r^2 + 2r,\ 3r^5 - 6r^3 + 3r.$

CHAPTER IX.

FRACTIONS.

135. An **algebraic fraction** is the indicated quotient of two expressions, written in the form $\frac{a}{b}$.

The dividend a is called the **numerator**, and the divisor b is called the **denominator.**

The numerator and denominator are called the **terms** of the fraction.

136. The introduction of the same factor into the dividend and divisor does not alter the value of the quotient, and the rejection of the same factor from the dividend and divisor does not alter the value of the quotient.

Thus $$\frac{12}{4}=3,\ \frac{2\times 12}{2\times 4}=3,\ \frac{12\div 2}{4\div 2}=3.$$

Hence, it follows, that

The value of a fraction is not altered if the numerator and denominator are both multiplied, or both divided, by the same factor.

Reduction of Fractions.

137. To reduce a fraction is to change its *form* without altering its *value.*

Case I.

138. To reduce a Fraction to its Lowest Terms.

A fraction is in its *lowest terms* when the numerator and denominator have no common factor. We have, therefore, the following rule:

Resolve the numerator and denominator into their prime factors, and cancel all the common factors; or, *divide the numerator and denominator by their highest common factor.*

Reduce the following fractions to their lowest terms:

(1) $\dfrac{38a^2b^3c^4}{57a^3bc^2} = \dfrac{2 \times 19a^2b^3c^4}{3 \times 19a^3bc^2} = \dfrac{2b^2c^2}{3a}.$

(2) $\dfrac{a^3 - x^3}{a^2 - x^2} = \dfrac{(a-x)(a^2+ax+x^2)}{(a-x)(a+x)} = \dfrac{a^2+ax+x^2}{a+x}.$

(3) $\dfrac{a^2+7a+10}{a^2+5a+6} = \dfrac{(a+5)(a+2)}{(a+3)(a+2)} = \dfrac{a+5}{a+3}.$

(4) $\dfrac{6x^2-5x-6}{8x^2-2x-15} = \dfrac{(2x-3)(3x+2)}{(2x-3)(4x+5)} = \dfrac{3x+2}{4x+5}.$

(5) $\dfrac{x^3-4x^2+4x-1}{x^3-2x^2+4x-3}.$

In example (5) we find by the method of division the H. C. F. of the numerator and denominator to be $x-1$.

The numerator divided by $x-1$ gives x^2-3x+1.

The denominator divided by $x-1$ gives x^2-x+3.

$\therefore \dfrac{x^3-4x^2+4x-1}{x^3-2x^2+4x-3} = \dfrac{x^2-3x+1}{x^2-x+3}.$

Exercise 46.

Reduce to lowest terms:

1. $\dfrac{x^2-1}{4x(x+1)}.$

2. $\dfrac{x^2-9x+20}{x^2-7x+12}.$

3. $\dfrac{x^2-2x-3}{x^2-10x+21}.$

4. $\dfrac{x^4+x^2+1}{x^2+x+1}.$

5. $\dfrac{x^6+2x^3y^3+y^6}{x^6-y^6}$.

6. $\dfrac{a^3+1}{a^3+2a^2+2a+1}$.

7. $\dfrac{a^2-a-20}{a^2+a-12}$.

8. $\dfrac{x^3-4x^2+9x-10}{x^3+2x^2-3x+20}$.

9. $\dfrac{x^3-5x^2+11x-15}{x^3-x^2+3x+5}$.

10. $\dfrac{x^4+x^3y+xy^3-y^4}{x^4-x^3y-xy^3-y^4}$.

11. $\dfrac{a^3+4a^2-5}{a^3-3a+2}$.

12. $\dfrac{3x^2+2x-1}{x^3+x^2-x-1}$.

13. $\dfrac{x^3-3x^2+4x-2}{x^3-x^2-2x+2}$.

14. $\dfrac{4x^2-12ax+9a^2}{8x^3-27a^3}$.

15. $\dfrac{15a^2+ab-2b^2}{9a^2+3ab-2b^2}$.

16. $\dfrac{a^2-b^2-2bc-c^2}{a^2+2ab+b^2-c^2}$.

17. $\dfrac{x^4-x^2-2x+2}{2x^3-x-1}$.

18. $\dfrac{x^3-6x^2+11x-6}{x^3-2x^2-x+2}$.

19. $\dfrac{6x^3-23x^2+16x-3}{6x^3-17x^2+11x-2}$.

20. $\dfrac{x^4-x^3-x+1}{x^4-2x^3-x^2-2x+1}$.

21. $\dfrac{a^5-a^4b-ab^4+b^5}{a^4-a^3b-a^2b^2+ab^3}$.

22. $\dfrac{(a+b)^2}{a^2-ab-2b^2}$.

23. $\dfrac{3ab(a^2-b^2)}{4(a^2b-ab^2)^2}$.

24. $\dfrac{a^2+2ab+b^2-c^2}{a^2+ab-ac}$.

25. $\dfrac{6x^3-11x^2y+3xy^2}{6x^2y-5xy^2-6y^3}$.

26. $\dfrac{a^2-(b+c+d)^2}{(a-b)^2-(c+d)^2}$.

27. $\dfrac{6x^2-5x-6}{8x^2-2x-15}$.

28. $\dfrac{x^4+x^2y^2+y^4}{(x-y)(x^3-y^3)}$.

29. $\dfrac{x^6+y^6}{x^4-x^2y^2+y^4}$.

30. $\dfrac{(a^3+b^3)(a^2+ab+b^2)}{(a^3-b^3)(a^2-ab+b^2)}$.

Case II.

139. To reduce a Fraction to an Integral or Mixed Expression

(1) Reduce $\frac{x^3-1}{x-1}$ to an integral expression.

$$\frac{x^3-1}{x-1}=x^2+x+1. \quad (\S\ 88)$$

(2) Reduce $\frac{x^3-1}{x+1}$ to a mixed expression.

$$\begin{array}{l|l}
x^3-1 & x+1 \\
\cline{2-2}
x^3+x^2 & x^2-x+1 \\
\cline{1-1}
\quad -x^2-1 & \\
\quad -x^2-x & \\
\cline{1-1}
\qquad\quad x-1 & \\
\qquad\quad x+1 & \\
\cline{1-1}
\qquad\quad\ -2 &
\end{array}$$

$$\therefore\ \frac{x^3-1}{x+1}=x^2-x+1-\frac{2}{x+1}.$$

Note. By the Law of Signs for division,

$$\frac{-2}{x+1} \text{ and } \frac{2}{-(x+1)}=-\frac{2}{x+1}.$$

The last form is the form usually written.

140. If the degree of the numerator of a fraction equals or exceeds that of the denominator, the fraction may be changed to a mixed or integral expression by the following rule:

Divide the numerator by the denominator.

Note. If there is a remainder, this remainder must be written as the numerator of a fraction of which the divisor is the denominator, and this fraction with its proper sign must be annexed to the integral part of the quotient.

Exercise 47.

Change to integral or mixed expressions:

1. $\dfrac{x^2-2x+1}{x-1}$.

2. $\dfrac{3x^2+2x+1}{x+4}$.

3. $\dfrac{3x^2+6x+5}{x+4}$.

4. $\dfrac{a^2-ax+x^2}{a+x}$.

5. $\dfrac{2x^2+5}{x-3}$.

6. $\dfrac{10a^2-17ax+10x^2}{5a-x}$.

7. $\dfrac{16(3x^2+1)}{4x-1}$.

8. $\dfrac{2x^2-5x-2}{x-4}$.

9. $\dfrac{a^2+b^2}{a-b}$.

10. $\dfrac{5x^3-x^2+5}{5x^2+4x-1}$.

Case III.

141. To reduce a Mixed Expression to a Fraction.

The process is precisely the same as in Arithmetic. Hence,

Multiply the integral expression by the denominator, to the product add the numerator, and under the result write the denominator.

Reduce to a fraction $a-b-\dfrac{a^2-ab-b^2}{a+b}$.

$$a-b-\frac{a^2-ab-b^2}{a+b}=\frac{(a-b)(a+b)-(a^2-ab-b^2)}{a+b}$$
$$=\frac{a^2-b^2-a^2+ab+b^2}{a+b}$$
$$=\frac{ab}{a+b}.$$

NOTE. The dividing line between the terms of a fraction has the force of a vinculum affecting the numerator. If, therefore, a *minus sign* precedes the dividing line, as in Example (2), and this line is

removed, the numerator of the given fraction must be inclosed in a parenthesis preceded by the minus sign, or the sign of every term of the numerator must be changed.

Exercise 48.

Change to fractional form:

1. $1-\frac{x-y}{x+y}$.
2. $1+\frac{x-y}{x+y}$.
3. $3x-\frac{1+2x^2}{x}$.
4. $a-x+\frac{a^2+x^2}{a-x}$.
5. $5a-2b-\frac{3a^2-4b^2}{5a-6b}$.
6. $a+b-\frac{a^2+b^2}{a+b}$.
7. $7a-\frac{2-3a+4a^2}{5-6a}$.
8. $3x-\frac{5ax-3}{2a}$.
9. $\frac{a+b}{a-b}+1$.
10. $\frac{a-b}{a+b}-1$.
11. $\frac{2x^2}{x+y}-(x+y)$.
12. $\frac{5a-12x}{4}+6a+3x$.
13. $a-1+\frac{1}{a+1}$.
14. $x+5-\frac{2x-15}{x-3}$.
15. $2a-b-\frac{2ab}{a+b}$.
16. $3x-10+\frac{41}{x+4}$.
17. $x^2+x+1+\frac{2}{x-1}$.
18. $x^3-3x-\frac{3x(3-x)}{x-2}$.
19. $a^2-2ax+4x^2-\frac{6x^3}{a+2x}$.
20. $x-a+y+\frac{a^2-ay+y^2}{x+a}$.

CASE IV.

142. To reduce Fractions to their Lowest Common Denominator. Since the value of a fraction is not altered by multiplying its numerator and denominator by the same factor (§ 136), any number of fractions can be reduced to equivalent fractions having the same denominator.

The process is the same as in Arithmetic. Hence we have the following rule:

Find the lowest common multiple of the denominators; this will be the required denominator. Divide this denominator by the denominator of each fraction.

Multiply the first numerator by the first quotient, the second numerator by the second quotient, and so on.

The products will be the respective numerators of the equivalent fractions.

NOTE. Every fraction should be in its lowest terms before the common denominator is found.

Reduce $\frac{1}{x^2+5x+6}$, $\frac{1}{x^2+2x+1}$ to equivalent fractions having the lowest common denominator.

$$\frac{1}{x^2+5x+6},\ \frac{1}{x^2+2x+1}$$
$$=\frac{1}{(x+3)(x+2)},\ \frac{1}{(x+1)(x+1)}.$$

$\therefore$ the lowest common denominator (L. C. D.) is

$$(x+3)(x+2)(x+1)^2.$$

The respective quotients are

$$(x+1)^2 \text{ and } (x+3)(x+2).$$

The respective products are

$$(x+1)^2 \text{ and } (x+3)(x+2).$$

Hence the required fractions are

$$\frac{(x+1)^2}{(x+3)(x+2)(x+1)^2} \text{ and } \frac{(x+3)(x+2)}{(x+3)(x+2)(x+1)^2}.$$

Exercise 49.

Express with lowest common denominator:

1. $\frac{3x-7}{6}, \frac{4x-9}{18}.$

2. $\frac{2x-4y}{5x^2}, \frac{3x-8y}{10x}.$

3. $\frac{4a-5c}{5ac}, \frac{3a-2c}{12a^2c}.$

4. $\frac{5}{1-x}, \frac{6}{1-x^2}.$

5. $\frac{1}{(a-b)(b-c)}, \frac{1}{(a-b)(a-c)}.$

6. $\frac{4x^2}{3(a+b)}, \frac{xy}{6(a^2-b^2)}.$

7. $\frac{8x+2}{x-2}, \frac{2x-1}{3x-6}, \frac{3x+2}{5x-10}.$

8. $\frac{a-bm}{mx}, 1, \frac{c-bn}{nx}.$

Addition and Subtraction of Fractions.

143. The algebraic sum of two or more fractions which have the same denominator is a fraction whose numerator is the algebraic sum of the numerators of the given fractions, and whose denominator is the common denominator of the given fractions. This follows from the distributive law of division.

If the fractions to be added have not the same denominator, they must first be reduced to equivalent fractions having the same denominator. (§ 142.)

Hence, to add fractions, we have the following rule:

Reduce the fractions to equivalent fractions having the same denominator; and write the sum of the numerators of these fractions over the common denominator.

Note. Each fraction should be expressed in its lowest terms.

144. When the Denominators are Simple Expressions.

Simplify $$\frac{3a-4b}{4}-\frac{2a-b+c}{3}+\frac{a-4c}{12}.$$

The L. C. D. = 12.

The multipliers, that is, the quotients obtained by dividing 12 by 4, 3, and 12, are 3, 4, and 1.

The products are

$$9a-12b,\ 8a-4b+4c, \text{ and } a-4c.$$

Hence the sum of the fractions equals

$$\frac{9a-12b}{12}-\frac{8a-4b+4c}{12}+\frac{a-4c}{12}$$

$$=\frac{9a-12b-(8a-4b+4c)+a-4c}{12}$$

$$=\frac{9a-12b-8a+4b-4c+a-4c}{12}$$

$$=\frac{2a-8b-8c}{12}$$

$$=\frac{a-4b-4c}{6}.$$

The above work may be arranged as follows:

The L. C. D. = 12.

The multipliers are 3, 4, and 1, respectively.

$$\begin{array}{lcll}
3(3a-4b) & = & 9a-12b & = \text{1st numerator.} \\
-4(2a-b+c) & = & -8a+\ 4b-4c & = \text{2d numerator.} \\
1(a-4c) & = & a \qquad\qquad -4c & = \text{3d numerator.} \\
\hline
 & & 2a-\ 8b-8c & \\
 & \text{or} & 2(a-4b-4c) & = \text{the sum of the numerators.}
\end{array}$$

$$\therefore \text{ sum of fractions} = \frac{2(a-4b-4c)}{12}=\frac{a-4b-4c}{6}.$$

Exercise 50.

Simplify:

1. $\dfrac{3x-2y}{5x}+\dfrac{5x-7y}{10x}+\dfrac{8x+2y}{25}.$

2. $\dfrac{4x^2-7y^2}{3x^2}+\dfrac{3x-8y}{6x}+\dfrac{5-2y}{12}.$

3. $\dfrac{4a^2+5b^2}{2b^2}+\dfrac{3a+2b}{5b}+\dfrac{7-2a}{9}.$

4. $\dfrac{4x+5}{3}-\dfrac{3x-7}{5x}+\dfrac{9}{12x^2}.$

5. $\dfrac{4x-3y}{7}+\dfrac{3x+7y}{14}-\dfrac{5x-2y}{21}+\dfrac{9x+2y}{42}.$

6. $\dfrac{3xy-4}{x^2y^2}-\dfrac{5y^2+7}{xy^3}-\dfrac{6x^2-11}{x^3y}.$

7. $\dfrac{a^2-2ac+c^2}{a^2c^2}-\dfrac{b^2-2bc+c^2}{b^2c^2}.$

8. $\dfrac{5a^3-2}{8a^2}-\dfrac{3a^2-a}{8}.$

9. $\dfrac{a-b}{c}+\dfrac{b-c}{a}+\dfrac{c-a}{b}+\dfrac{ab^2+bc^2+ca^2}{abc}.$

10. $\dfrac{1}{2x^2y}-\dfrac{1}{6y^2z}-\dfrac{1}{2xz^2}+\dfrac{2x-z}{4x^2z^2}+\dfrac{y-2z}{4x^2yz}.$

145. When the Denominators have Compound Expressions.

(1) Simplify $\dfrac{2a+b}{a-b}-\dfrac{2a-b}{a+b}-\dfrac{6ab}{a^2-b^2}$.

The L. C. D. is $(a-b)(a+b)$.
The multipliers are $a+b$, $a-b$, and 1, respectively.

$$\begin{aligned}
(a+b)(2a+b) &= \quad 2a^2+3ab+b^2 = \text{1st numerator.}\\
-(a-b)(2a-b) &= -2a^2+3ab-b^2 = \text{2d numerator.}\\
-1(6ab) &= \qquad -6ab \qquad = \text{3d numerator.}\\
&\qquad\qquad 0 \qquad\qquad = \text{sum of numerators.}
\end{aligned}$$

$\therefore$ sum of fractions $=0$.

(2) Simplify $\dfrac{x-1}{x-2}+\dfrac{x-2}{x-3}+\dfrac{x-3}{x-4}$.

The L. C. D. is $(x-2)(x-3)(x-4)$.

$$\begin{aligned}
(x-1)(x-3)(x-4) &= x^3-8x^2+19x-12 = \text{1st numerator.}\\
(x-2)(x-2)(x-4) &= x^3-8x^2+20x-16 = \text{2d numerator.}\\
(x-2)(x-3)(x-3) &= x^3-8x^2+21x-18 = \text{3d numerator.}\\
&\quad 3x^3-24x^2+60x-46 = \text{sum of numerators.}
\end{aligned}$$

$\therefore$ sum of fractions $=\dfrac{3x^3-24x^2+60x-46}{(x-2)(x-3)(x-4)}$.

Exercise 51.

Simplify :

1. $\dfrac{1}{x-6}+\dfrac{1}{x+5}$.
2. $\dfrac{1}{x-7}-\dfrac{1}{x-3}$.
3. $\dfrac{1}{1+x}+\dfrac{1}{1-x}$.
4. $\dfrac{1}{1-x}-\dfrac{2}{1-x^2}$.
5. $\dfrac{1}{x-y}+\dfrac{x}{(x-y)^2}$.
6. $\dfrac{1}{2a(a+x)}+\dfrac{1}{2a(a-x)}$.
7. $\dfrac{a}{(a+b)b}-\dfrac{b}{(a-b)a}$.
8. $\dfrac{5}{2x(x-1)}-\dfrac{3}{4x(x-2)}$.
9. $\dfrac{1+x}{1+x+x^2}-\dfrac{1-x}{1-x+x^2}$.
10. $\dfrac{2ax-3by}{2xy(x-y)}-\dfrac{2ax+3by}{2xy(x+y)}$.

Exercise 52.

Simplify :

1. $\dfrac{1}{1+a}+\dfrac{1}{1-a}+\dfrac{2a}{1-a^2}$.

3. $\dfrac{x}{1-x}-\dfrac{x^2}{1-x}+\dfrac{x}{1+x^2}$.

2. $\dfrac{1}{1-x}-\dfrac{1}{1+x}+\dfrac{2x}{1+x^2}$.

4. $\dfrac{x}{y}+\dfrac{y}{x+y}+\dfrac{x^2}{x^2+xy}$.

5. $\dfrac{x-1}{x-2}+\dfrac{x-2}{x-3}+\dfrac{x-3}{x-4}$.

6. $\dfrac{3}{x-a}+\dfrac{4a}{(x-a)^2}-\dfrac{5a^2}{(x-a)^3}$.

7. $\dfrac{1}{x-1}-\dfrac{1}{x+2}-\dfrac{3}{(x+1)(x+2)}$.

8. $\dfrac{a-b}{(b+c)(c+a)}+\dfrac{b-c}{(c+a)(a+b)}+\dfrac{c-a}{(a+b)(b+c)}$.

9. $\dfrac{x-a}{x-b}+\dfrac{x-b}{x-a}-\dfrac{(a-b)^2}{(x-a)(x-b)}$.

10. $\dfrac{x+y}{y}-\dfrac{2x}{x+y}+\dfrac{x^2y-x^3}{y(x^2-y^2)}$.

11. $\dfrac{a+b}{(b-c)(c-a)}+\dfrac{b+c}{(c-a)(a-b)}+\dfrac{c+a}{(a-b)(b-c)}$.

12. $\dfrac{a^2-bc}{(a+b)(a+c)}+\dfrac{b^2-ac}{(b+a)(b+c)}+\dfrac{c^2+ab}{(c+b)(c+a)}$.

13. $\dfrac{a}{a-x}-\dfrac{x}{a+2x}-\dfrac{a^2+x^2}{(a-x)(a+2x)}$.

14. $\dfrac{3}{(a-b)(b-c)}-\dfrac{4}{(a-b)(a-c)}+\dfrac{6}{(a-c)(b-c)}$.

15. $\dfrac{x-2y}{x(x-y)}-\dfrac{2x+y}{y(x+y)}-\dfrac{2x}{x^2-y^2}$.

16. $\dfrac{a-b}{x(a+b)}-\dfrac{a-b}{y(a+b)}-\dfrac{(a-b)(x+y)}{xy(a+b)}$.

17. $\dfrac{3x}{(x+y)^2}-\dfrac{x+2y}{x^2-y^2}+\dfrac{3y}{(x-y)^2}$.

18. $\dfrac{a-c}{(a+b)^2-c^2}-\dfrac{a-b}{(a+c)^2-b^2}$.

19. $\dfrac{a+b}{ax+by}-\dfrac{a-b}{ax-by}+\dfrac{ab(x-y)}{a^2x^2-b^2y^2}$.

146. When the terms of the denominators are not arranged in the same order.

Since $\dfrac{ab}{b}=a$, and $\dfrac{-ab}{-b}=a$, it follows that

The value of a fraction is not altered if the signs of the numerator and denominator are both changed.

It follows, also, by the Law of Signs, that

The value of a fraction is not altered if the signs of *any even number of factors* in the numerator and denominator of a fraction are changed.

147. Since changing the sign before a fraction is equivalent to changing the sign before the numerator or the denominator, it follows that

The sign before the denominator may be changed, provided the sign before the fraction is changed.

Note. If the denominator is a compound expression, the beginner must remember that the sign of the denominator is changed by changing the sign of every term of the denominator. Thus,

$$\frac{x}{a-x}=-\frac{x}{x-a}.$$

(1) Simplify $\dfrac{2}{x}-\dfrac{3}{2x-1}+\dfrac{2x-3}{1-4x^2}$.

Changing the signs before the terms of the denominator of the third fraction, and the sign before the fraction, we have

$$\frac{2}{x}-\frac{3}{2x-1}-\frac{2x-3}{4x^2-1}.$$

The L. C. D. $=x(2x-1)(2x+1)$.

$$\begin{aligned}
2(2x-1)(2x+1) &= 8x^2-2 &&= \text{1st numerator.}\\
-3x(2x+1) &= -6x^2-3x &&= \text{2d numerator.}\\
-x(2x-3) &= -2x^2+3x &&= \text{3d numerator.}\\
\hline
& \quad -2 &&= \text{sum of numerators.}
\end{aligned}$$

$\therefore$ sum of the fractions $=-\dfrac{2}{x(2x-1)(2x+1)}$.

(2) Simplify

$$\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-a)(b-c)}+\frac{1}{c(c-a)(c-b)}.$$

NOTE. Change the sign of the factor $(b-a)$ in the denominator of the second fraction, and change the sign before the fraction.

Change the signs of the two factors $(c-a)$ and $(c-b)$ in the denominator of the third fraction. We now have

$$\frac{1}{a(a-b)(a-c)}-\frac{1}{b(a-b)(b-c)}+\frac{1}{c(a-c)(b-c)}.$$

The L. C. D. $=abc(a-b)(a-c)(b-c)$.

$$\begin{aligned}
bc(b-c) &= b^2c-bc^2 &&= \text{1st numerator.}\\
-ac(a-c) &= -a^2c+ac^2 &&= \text{2d numerator.}\\
ab(a-b) &= a^2b-ab^2 &&= \text{3d numerator.}\\
\hline
a^2b-a^2c-ab^2+ac^2+b^2c-bc^2 &&&= \text{sum of numerators.}
\end{aligned}$$

$$\begin{aligned}
&= a^2(b-c)-a(b^2-c^2)+bc(b-c),\\
&= [a^2-a(b+c)+bc][b-c],\\
&= [a^2-ab-ac+bc][b-c],\\
&= [(a^2-ac)-(ab-bc)][b-c],\\
&= [a(a-c)-b(a-c)][b-c],\\
&= (a-b)(a-c)(b-c).
\end{aligned}$$

$\therefore$ sum of the fractions $=\dfrac{(a-b)(a-c)(b-c)}{abc(a-b)(a-c)(b-c)}=\dfrac{1}{abc}$.

Exercise 53.

Simplify:

1. $\dfrac{x}{x-y}+\dfrac{x-y}{y-x}$.

2. $\dfrac{3+2x}{2-x}+\dfrac{3x-2}{2+x}+\dfrac{16x-x^2}{x^2-4}$.

3. $\dfrac{x^2}{x^2-1}+\dfrac{x}{x+1}-\dfrac{x}{1-x}$.

4. $\dfrac{4}{3-3y^2}+\dfrac{1}{2-2y}+\dfrac{1}{6y+6}$.

5. $\dfrac{1}{(2-m)(3-m)}-\dfrac{2}{(m-1)(m-3)}+\dfrac{1}{(m-1)(m-2)}$.

6. $\dfrac{1}{(b-a)(x+a)}+\dfrac{1}{(a-b)(x+b)}$.

7. $\dfrac{a^2+b^2}{a^2-b^2}+\dfrac{2ab^2}{b^3-a^3}+\dfrac{2a^2b}{a^3+b^3}$.

8. $\dfrac{b-a}{x-b}-\dfrac{a-2b}{b+x}-\dfrac{3x(a-b)}{b^2-x^2}$.

9. $\dfrac{3+2x}{2-x}-\dfrac{2-3x}{2+x}+\dfrac{16x-x^2}{x^2-4}$.

10. $\dfrac{3}{1-2x}-\dfrac{7}{1+2x}-\dfrac{4-20x}{4x^2-1}$.

11. $\dfrac{a+b}{(b-c)(c-a)}+\dfrac{b+c}{(b-a)(a-c)}+\dfrac{c+a}{(a-b)(b-c)}$.

12. $\dfrac{a^2-bc}{(a-b)(a-c)}+\dfrac{b^2+ac}{(b+c)(b-a)}+\dfrac{c^2+ab}{(c-a)(c+b)}$.

13. $\dfrac{y+z}{(x-y)(x-z)}+\dfrac{z+x}{(y-x)(y-z)}+\dfrac{x+y}{(z-x)(z-y)}.$

14. $\dfrac{3}{(a-b)(b-c)}-\dfrac{4}{(b-a)(c-a)}-\dfrac{6}{(a-c)(c-b)}.$

15. $\dfrac{1}{x(x-y)(x-z)}+\dfrac{1}{y(y-x)(y-z)}-\dfrac{1}{xyz}.$

Multiplication and Division of Fractions.

148. Multiplication of Fractions.

The expression $\frac{a}{b}\times\frac{c}{d}$ means that we are to multiply the quotient $\frac{c}{d}$ by a, and divide the result by b.

From the nature of division, if we multiply the dividend c by a, we multiply the quotient $\frac{c}{d}$ by a, and obtain $\frac{ac}{d}$; if we multiply the divisor d by b, we divide the quotient $\frac{ac}{d}$ by b, and obtain $\frac{ac}{bd}$. Hence,

$$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}.$$

Therefore, to find the product of two fractions,

Find the product of the numerators for the required numerator, and the product of the denominators for the required denominator.

In like manner,

$$\frac{a}{b}\times\frac{c}{d}\times\frac{e}{f}=\frac{ac}{bd}\times\frac{e}{f}=\frac{ace}{bdf};$$

and so on for any number of fractions.

Again, $$\left(\frac{a}{b}\right)^2 = \frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}.$$

In like manner,

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

149. Division of Fractions. If the product of two numbers is equal to 1, each of the numbers is called the **reciprocal** of the other.

The reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$,

for $$\frac{b}{a} \times \frac{a}{b} = \frac{ba}{ab} = 1.$$

The reciprocal of a fraction, therefore, is the fraction inverted.

Since $$\frac{a}{b} \div \frac{a}{b} = 1,$$

and $$\frac{b}{a} \times \frac{a}{b} = 1,$$ it follows that

To divide by a fraction is the same as to multiply by its reciprocal.

To divide by a fraction, therefore,

Invert the divisor and multiply.

NOTE. Every mixed expression should first be reduced to a fraction, and every integral expression should be written as a fraction having 1 for the denominator. If a factor is common to a numerator and a denominator, it should be cancelled, as the cancelling of a common factor *before* the multiplication is evidently equivalent to cancelling it *after* the multiplication.

(1) Find the product of $\frac{2a^2b}{3cd^2} \times \frac{6c^2d}{5ab} \times \frac{5ab^2c}{8a^2c^2d^2}$.

$$\frac{2a^2b}{3cd^2} \times \frac{6c^2d}{5ab} \times \frac{5ab^2c}{8a^2c^2d^2} = \frac{2 \times 6 \times 5a^3b^3c^3d}{3 \times 5 \times 8a^3bc^3d^4} = \frac{b^2}{2d^3}.$$

(2) Find the product of

$$\frac{x^2 - y^2}{x^2 - 3xy + 2y^2} \times \frac{xy - 2y^2}{x^2 + xy} \times \frac{x^2 - xy}{(x - y)^2}.$$

$$\begin{aligned}\frac{x^2 - y^2}{x^2 - 3xy + 2y^2} \times \frac{xy - 2y^2}{x^2 + xy} \times \frac{x^2 - xy}{(x - y)^2} \\ &= \frac{(x-y)(x+y)}{(x-y)(x-2y)} \times \frac{y(x-2y)}{x(x+y)} \times \frac{x(x-y)}{(x-y)(x-y)} \\ &= \frac{y}{x-y}.\end{aligned}$$

NOTE. The common factors cancelled are $x - y$, $x + y$, $x - 2y$, x, and $x - y$.

(3) Find the quotient of $\frac{ax}{(a - x)^2} \div \frac{ab}{a^2 - x^2}$.

$$\begin{aligned}\frac{ax}{(a-x)^2} \div \frac{ab}{a^2 - x^2} &= \frac{ax}{(a-x)(a-x)} \times \frac{(a-x)(a+x)}{ab} \\ &= \frac{x(a+x)}{b(a-x)}.\end{aligned}$$

The common factors cancelled are a and $a - x$.

Exercise 54.

Simplify:

1. $\frac{a}{bx} \times \frac{cx}{d}$.

2. $\frac{2x}{a} \times \frac{3ab}{c} \times \frac{3ac}{2b}$.

3. $\frac{3p}{2p - 2} \div \frac{2p}{p - 1}$.

4. $\frac{8x^4y}{15ab^3} \div \frac{2x^3}{3ab^2}$.

5. $\dfrac{8a^2b^3}{45x^2y} \times \dfrac{15xy^2}{24a^3b^2}$.

6. $\dfrac{9x^2y^2z}{10a^2b^2c} \times -\dfrac{20a^3b^2c}{18xy^2z}$.

7. $\dfrac{3x^2y}{4xz^2} \times \dfrac{5y^2z}{6xy} \times -\dfrac{12x^2}{2xy^2}$.

8. $\dfrac{9m^2n^2}{8p^3q^3} \times \dfrac{5p^2q}{2xy} \times \dfrac{24x^2y^2}{90mn}$.

9. $\dfrac{25k^3m^2}{14n^2q^2} \times \dfrac{70n^3q}{75p^2m} \times \dfrac{3pm}{4k^2n}$.

10. $\dfrac{a-b}{a^2+ab} \times \dfrac{a^2-b^2}{a^2-ab}$.

11. $\dfrac{a^2+b^2}{a^2-b^2} \div \dfrac{a-b}{a+b}$.

12. $\dfrac{x^2+x-2}{x^2-7x} \times \dfrac{x^2-13x+42}{x^2+2x}$.

13. $\dfrac{x^2-11x+30}{x^2-6x+9} \times \dfrac{x^2-3x}{x^2-5x}$.

14. $\dfrac{a^3-x^3}{a^3+x^3} \times \dfrac{(a+x)^2}{(a-x)^2}$.

15. $\dfrac{2a(x^2-y^2)^2}{cx} \times \dfrac{x^3}{(x-y)(x+y)^2}$.

16. $\dfrac{a^2+2ab}{a^2+4b^2} \times \dfrac{ab-2b^2}{a^2-4b^2}$.

17. $\dfrac{x^2-4}{x^2+5x} \times \dfrac{x^2-25}{x^2+2x}$.

18. $\dfrac{x^2+xy}{x-y} \times \dfrac{(x-y)^2}{x^4-y^4}$.

19. $\dfrac{m^2-n^2}{c^3+d^3} \div \dfrac{n-m}{c+d}$.

20. $\dfrac{a^2-4a+3}{a^2-5a+4} \times \dfrac{a^2-9a+20}{a^2-10a+21} \times \dfrac{a^2-7a}{a^2-5a}$.

21. $\dfrac{b^2-7b+6}{b^2+3b-4} \times \dfrac{b^2+10b+24}{b^2-14b+48} \div \dfrac{b^2+6b}{b^3-8b^2}$.

22. $\dfrac{x^2-y^2}{x^2-3xy+2y^2} \times \dfrac{xy-2y^2}{x^2+xy} \times \dfrac{x^2-xy}{(x-y)^2}$.

23. $\dfrac{a^3-3a^2b+3ab^2-b^3}{a^2-b^2} \div \dfrac{2ab-2b^2}{3} \times \dfrac{a^2+ab}{a-b}$.

24. $\dfrac{(a+b)^2-c^2}{a^2-(b-c)^2} \div \dfrac{c^2-(a+b)^2}{c^2-(a-b)^2}.$

25. $\dfrac{(x-a)^2-b^2}{(x-b)^2-a^2} \times \dfrac{x^2-(b-a)^2}{x^2-(a-b)^2}.$

26. $\dfrac{(a+b)^2-(c+d)^2}{(a+c)^2-(b+d)^2} \div \dfrac{(a-c)^2-(d-b)^2}{(a-b)^2-(d-c)^2}.$

27. $\dfrac{x^2-2xy+y^2-z^2}{x^2+2xy+y^2-z^2} \times \dfrac{x+y-z}{x-y+z}.$

150. Complex Fractions. A complex fraction is one that has a fraction in the numerator, or in the denominator, or in both.

NOTE. Generally, the shortest way to simplify a complex fraction is to multiply both terms of the fraction by the L. C. D. of the fractions contained in the numerator and denominator.

(1) Simplify $\dfrac{\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}}{\dfrac{a+x}{a-x}+\dfrac{a-x}{a+x}}.$

The L. C. D. of the fractions in the numerator and denominator is

$$(a-x)(a+x).$$

Multiply by $(a-x)(a+x)$, and the result is

$$\frac{(a+x)^2-(a-x)^2}{(a+x)^2+(a-x)^2}$$
$$=\frac{(a^2+2ax+x^2)-(a^2-2ax+x^2)}{(a^2+2ax+x^2)+(a^2-2ax+x^2)}$$
$$=\frac{a^2+2ax+x^2-a^2+2ax-x^2}{a^2+2ax+x^2+a^2-2ax+x^2}$$
$$=\frac{4ax}{2a^2+2x^2}$$
$$=\frac{2ax}{a^2+x^2}.$$

(2) Simplify $\dfrac{x}{1-\dfrac{x}{1+x+\dfrac{x}{1-x+x^2}}}$.

$$\frac{x}{1-\dfrac{x}{1+x+\dfrac{x}{1-x+x^2}}}=\frac{x}{1-\dfrac{x(1-x+x^2)}{(1+x)(1-x+x^2)+x}}$$

$$=\frac{x}{1-\dfrac{x-x^2+x^3}{1+x+x^3}}$$

$$=\frac{x(1+x+x^3)}{1+x+x^3-(x-x^2+x^3)}$$

$$=\frac{x+x^2+x^4}{1+x^2}.$$

NOTE. In a fraction of this kind, called a *continued fraction*, we begin at the *bottom*, and reduce step by step. Thus, in the last example, we take out the fraction $\dfrac{x}{1+x+\dfrac{x}{1-x+x^2}}$, and multiply the numerator and denominator by $1-x+x^2$, getting for the result, $\dfrac{x(1-x+x^2)}{(1+x)(1-x+x^2)+x}$, which simplified is $\dfrac{x-x^2+x^3}{1+x+x^3}$.

Putting this fraction in the given complex fraction for

$$\frac{x}{1+x+\dfrac{x}{1-x+x^2}},$$

we have
$$\frac{x}{1-\dfrac{x-x^2+x^3}{1+x+x^3}}.$$

Multiplying both terms by $1+x+x^3$, we get

$$\frac{x(1+x+x^3)}{1+x+x^3-x+x^2-x^3}$$

$$=\frac{x+x^2+x^4}{1+x^2}.$$

Exercise 55.

Simplify:

1. $$\frac{\dfrac{3x}{2}+\dfrac{x-1}{3}}{\dfrac{13}{6}(x+1)-\dfrac{x}{3}-2\frac{1}{2}}.$$

2. $$\frac{x-1+\dfrac{6}{x-6}}{x-2+\dfrac{3}{x-6}}.$$

3. $$\frac{3}{x+1}-\frac{2x-1}{x^2+\dfrac{x}{2}-\dfrac{1}{2}}.$$

4. $$\frac{x-a}{x-\dfrac{(x-b)(x-c)}{x+a}}.$$

5. $$\frac{\left(\dfrac{a}{x}-\dfrac{x}{a}\right)\left(\dfrac{a}{x}+\dfrac{x}{a}\right)}{1-\dfrac{x-a}{x+a}}.$$

6. $$\frac{\dfrac{1}{x-y}-\dfrac{x}{x^2-y^2}}{\dfrac{x}{xy+y^2}-\dfrac{y}{x^2+xy}}.$$

7. $$\frac{\dfrac{x+1}{x-1}+\dfrac{x-1}{x+1}}{\dfrac{x+1}{x-1}-\dfrac{x-1}{x+1}}.$$

8. $$1-\frac{1}{1+\dfrac{1}{x}}.$$

9. $$1+\frac{x}{1+x+\dfrac{2x^2}{1-x}}.$$

10. $$\frac{1}{1-\dfrac{1}{1+\dfrac{1}{x}}}.$$

11. $$\frac{1}{1+\dfrac{x}{1+x+\dfrac{2x^2}{1-x}}}.$$

12. $$\frac{\left(\dfrac{a}{x}+\dfrac{x}{a}-2\right)\left(\dfrac{a}{x}+\dfrac{x}{a}+2\right)}{\left(\dfrac{a}{x}-\dfrac{x}{a}\right)^2}.$$

13. $$\frac{\dfrac{x^2+y^2}{x^2-y^2}+\dfrac{2x}{x+y}\left\{\dfrac{xy-x^2}{(x-y)^2}+\dfrac{x+y}{x-y}\right\}}{x-y}.$$

14. $$\frac{\dfrac{(x^2-y^2)(2x^2-2xy)}{4(x-y)^2}}{\dfrac{xy}{x+y}}.$$

15. $$\frac{\dfrac{ab}{x^2+(a+b)x+ab}-\dfrac{ac}{x^2+(a+c)x+ac}}{\dfrac{b-c}{x^2+(b+c)x+bc}}.$$

16. $$\frac{x}{1+\dfrac{1}{x}}+1-\frac{1}{x+1}.$$

17. $$\frac{\dfrac{a+b}{b}+\dfrac{b}{a+b}}{\dfrac{1}{a}+\dfrac{1}{b}}.$$

18. $$\frac{2m-3+\dfrac{1}{m}}{\dfrac{2m-1}{m}}.$$

19. $$\frac{\dfrac{1}{ab}+\dfrac{1}{ac}+\dfrac{1}{bc}}{\dfrac{a^2-(b+c)^2}{ab}}.$$

20. $$\frac{3}{1+\dfrac{3}{1+\dfrac{3}{1-x}}}.$$

Exercise 56.

MISCELLANEOUS EXAMPLES.

1. Simplify $\dfrac{x^4-9x^3+7x^2+9x-8}{x^4+7x^3-9x^2-7x+8}$.

2. Find the value of $\dfrac{a^2+b^2-c^2+2ab}{a^2-b^2-c^2+2bc}$ when $a=4$, $b=\frac{1}{2}$, $c=1$.

3. Find the value of $3a^2+\dfrac{2ab^2}{c}-\dfrac{c^3}{b^2}$ when $a=4$, $b=\frac{1}{2}$, $c=1$.

4. Simplify $\dfrac{2}{(x^2-1)^2}-\dfrac{1}{2x^2-4x+2}-\dfrac{1}{1-x^2}$.

5. Simplify $\left(\dfrac{x}{1+\dfrac{1}{x}}+1-\dfrac{1}{x+1}\right)\div\left(\dfrac{x}{1-\dfrac{1}{x}}-x-\dfrac{1}{x-1}\right)$.

6. Find the value of $\left(\dfrac{x-a}{x-b}\right)^3-\dfrac{x-2a+b}{x+a-2b}$ when $x=\dfrac{a+b}{2}$.

7. Simplify $\left\{\dfrac{a+b}{2(a-b)}-\dfrac{a-b}{2(a+b)}+\dfrac{2b^2}{a^2-b^2}\right\}\dfrac{a-b}{2b}$.

8. Simplify $\left(\frac{x^2+y^2}{x^2-y^2}-\frac{x^2-y^2}{x^2+y^2}\right)\div\left(\frac{x+y}{x-y}-\frac{x-y}{x+y}\right)$.

9. Simplify

$$\left(\frac{x^2}{y^2}-1\right)\left(\frac{x}{x-y}-1\right)+\left(\frac{x^3}{y^3}-1\right)\left(\frac{x^2+xy}{x^2+xy+y^2}-1\right).$$

10. Simplify

$$\left(\frac{a^2-ab}{a^3-b^3}\right)\left(\frac{a^2+ab+b^2}{a+b}\right)+\left(\frac{2a^3}{a^3+b^3}-1\right)\left(1-\frac{2ab}{a^2+ab+b^2}\right).$$

11. Simplify $\dfrac{1+\dfrac{a-x}{a+x}}{1-\dfrac{a-x}{a+x}}\div\dfrac{1+\dfrac{a^2-x^2}{a^2+x^2}}{1-\dfrac{a^2-x^2}{a^2+x^2}}$.

12. Divide $x^3+\frac{1}{x^3}-3\left(\frac{1}{x^2}-x^2\right)+4\left(x+\frac{1}{x}\right)$ by $x+\frac{1}{x}$.

13. Simplify $\dfrac{1-\dfrac{2xy}{(x+y)^2}}{1+\dfrac{2xy}{(x-y)^2}}\div\left\{\dfrac{1-\dfrac{y}{x}}{1+\dfrac{y}{x}}\right\}^2$.

14. Find the value of $\frac{x+2a}{2b-x}+\frac{x-2a}{2b+x}-\frac{4ab}{4b^2-x^2}$ when $x=\frac{ab}{a+b}$.

15. Find the value of $\frac{x+y-1}{x-y+1}$ when $x=\frac{a+1}{ab+1}$ and $y=\frac{ab+a}{ab+1}$.

16. Simplify

$$\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-c)(b-a)}+\frac{1}{c(c-a)(c-b)}.$$

17. Simplify $\dfrac{3abc}{bc+ca-ab}-\dfrac{\dfrac{a-1}{a}+\dfrac{b-1}{b}+\dfrac{c-1}{c}}{\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{1}{c}}$.

18. Simplify $\dfrac{\dfrac{m^2+n^2}{n}-m}{\dfrac{1}{n}-\dfrac{1}{m}} \times \dfrac{m^2-n^2}{m^3+n^3}$.

19. Simplify $\dfrac{\dfrac{1}{a}+\dfrac{1}{b+c}}{\dfrac{1}{a}-\dfrac{1}{b+c}}\left\{1+\dfrac{b^2+c^2-a^2}{2bc}\right\}$.

20. Simplify $3a-[b+\{2a-(b-c)\}]+\dfrac{1}{2}+\dfrac{2c^2-\frac{1}{2}}{2c+1}$.

21. Simplify $\dfrac{\dfrac{1}{a-x}-\dfrac{1}{a-y}+\dfrac{x}{(a-x)^2}-\dfrac{y}{(a-y)^2}}{\dfrac{1}{(a-y)(a-x)^2}-\dfrac{1}{(a-x)(a-y)^2}}$.

22. Simplify $\dfrac{1}{x+\dfrac{1}{1+\dfrac{x+1}{3-x}}}$.

23. $\dfrac{(x^2-y^2)(2x^2-2xy)}{4(x-y)^2\div\dfrac{xy}{x+y}}$.

24. Simplify $\left(\dfrac{c-b}{c+b}-\dfrac{c^3-b^3}{c^3+b^3}\right)\div\left(\dfrac{c+b}{c-b}+\dfrac{c^2+b^2}{c^2-b^2}\right)$.

25. Simplify $\dfrac{y}{(x-y)(x-z)}+\dfrac{x}{(y-x)(y-z)}+\dfrac{x+y}{(z-x)(z-y)}$.

26. Simplify $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-a)(b-c)}-\dfrac{1}{abc}$.

27. Simplify $\dfrac{x-4+\dfrac{6}{x+1}}{x-\dfrac{6}{x-1}}\times\dfrac{1-\dfrac{x+5}{x^2-1}}{(x-1)(x-2)}$.

CHAPTER X.

FRACTIONAL EQUATIONS.

151. To reduce Equations containing Fractions.

(1) Solve $\frac{x}{3} - \frac{x-1}{11} = x - 9.$

Multiply by 33, the L. C. M. of the denominators.

Then,
$$11x - 3x + 3 = 33x - 297,$$
$$11x - 3x - 33x = -297 - 3,$$
$$-25x = -300.$$
$$\therefore x = 12.$$

NOTE. Since the minus sign precedes the second fraction, in removing the denominator, the + (understood) before x, the first term of the numerator, is changed to −, and the − before 1, the second term of the numerator, is changed to +.

Therefore, to clear an equation of fractions,

Multiply each term by the L. C. M. of the denominators.

If a fraction is preceded by a **minus sign,** *the sign of every term of the numerator must be changed when the denominator is removed.*

(2) Solve $\frac{x-4}{x-5} - \frac{x-5}{x-6} = \frac{x-7}{x-8} - \frac{x-8}{x-9}.$

NOTE. The solution of this and similar problems will be much easier by combining the fractions on the left side and the fractions on the right side than by the rule given above.

$$\frac{(x-4)(x-6)-(x-5)^2}{(x-5)(x-6)} = \frac{(x-7)(x-9)-(x-8)^2}{(x-8)(x-9)}.$$

By simplifying the numerators, we have

$$\frac{-1}{(x-5)(x-6)}=\frac{-1}{(x-8)(x-9)}.$$

Since the numerators are equal, the denominators are equal.

Hence, $(x-5)(x-6)=(x-8)(x-9)$.

Solving, we have $x=7$.

Exercise 57.

Solve:

1. $5x-\frac{x+2}{2}=71.$
2. $x-\frac{3-x}{3}=\frac{17}{3}.$
3. $\frac{5-2x}{4}+2=x-\frac{6x-8}{2}.$
4. $\frac{5x}{2}-\frac{5x}{4}=\frac{9}{4}-\frac{3-x}{2}.$
5. $2x-\frac{5x-4}{6}=7-\frac{1-2x}{5}.$
6. $\frac{x+2}{2}=\frac{14}{9}-\frac{3+5x}{4}.$
7. $\frac{5x+3}{8}-\frac{3-4x}{3}+\frac{x}{2}=\frac{31}{2}-\frac{9-5x}{6}.$
8. $\frac{10x+3}{3}-\frac{6x-7}{2}=10(x-1).$
9. $\frac{5x-7}{2}-\frac{2x+7}{3}=3x-14.$
10. $\frac{7x+5}{6}-\frac{5x-6}{4}=\frac{8-5x}{12}.$
11. $\frac{x+4}{3}-\frac{x-4}{5}=2+\frac{3x-1}{15}.$
12. $\frac{3x+5}{7}-\frac{2x+7}{3}+10-\frac{3x}{5}=0.$
13. $\frac{1}{7}(3x-4)+\frac{1}{3}(5x+3)=43-5x.$
14. $\frac{1}{2}(27-2x)=\frac{9}{2}-\frac{1}{10}(7x-54).$

15. $5x-\{8x-3[16-6x-(4-5x)]\}=6.$

16. $\frac{5x-3}{7}-\frac{9-x}{3}=\frac{5x}{2}+\frac{19}{6}(x-4).$

17. $\frac{2x+7}{7}-\frac{9x-8}{11}=\frac{x-11}{2}.$

18. $\frac{8x-15}{3}-\frac{11x-1}{7}=\frac{7x+2}{13}.$

19. $\frac{7x+9}{8}-\frac{3x+1}{7}=\frac{9x-13}{4}-\frac{249-9x}{14}.$

152. If the denominators contain both simple and compound expressions, it is best to remove the simple expressions first, and then each compound expression in turn. After each multiplication the result should be reduced to the simplest form.

(1) Solve $\frac{8x+5}{14}+\frac{7x-3}{6x+2}=\frac{4x+6}{7}.$

Multiply both sides by 14.

Then, $$8x+5+\frac{7(7x-3)}{3x+1}=8x+12.$$

Transpose and combine, $\frac{7(7x-3)}{3x+1}=7.$

Divide by 7 and multiply by $3x+1$,

$$7x-3=3x+1.$$

$$\therefore x=1.$$

(2) Solve $\dfrac{3-\dfrac{4x}{9}}{4}=\dfrac{1}{4}-\dfrac{\dfrac{7x}{9}-3}{10}.$

Multiply both terms of each complex fraction by 9.

Then, $$\frac{27-4x}{36}=\frac{1}{4}-\frac{7x-27}{90}.$$

Solving this equation, we have $x=6$.

Exercise 58.

Solve the equations:

1. $\frac{9x+20}{36}=\frac{4(x-3)}{5x-4}+\frac{x}{4}.$

2. $\frac{9(2x-3)}{14}+\frac{11x-1}{3x+1}=\frac{9x+11}{7}.$

3. $\frac{10x+17}{18}-\frac{12x+2}{13x-16}=\frac{5x-4}{9}.$

4. $\frac{6x+13}{15}-\frac{3x+5}{5x-25}=\frac{2x}{5}.$

5. $\frac{18x-22}{39-6x}+2x+\frac{1+16x}{24}=4\tfrac{5}{12}-\frac{101-64x}{24}.$

6. $\frac{6-5x}{15}-\frac{7-2x^2}{14(x-1)}=\frac{1+3x}{21}-\frac{10x-11}{30}+\frac{1}{105}.$

7. $\frac{9x+5}{14}+\frac{8x-7}{6x+2}=\frac{36x+15}{56}+\frac{41}{56}.$

8. $\frac{6x+7}{15}-\frac{2x-2}{7x-6}=\frac{2x+1}{5}.$

9. $\frac{6x+1}{15}-\frac{2x-4}{7x-16}=\frac{2x-1}{5}.$

10. $\frac{7x-6}{35}-\frac{x-5}{6x-101}=\frac{x}{5}.$

153. Literal equations are equations in which some or all of the given numbers are represented by letters; the *first* letters of the alphabet are used to represent known numbers.

(1) $(a-x)(a+x)=2a^2+2ax-x^2.$

$$\text{Then, } a^2-x^2=2a^2+2ax-x^2,$$
$$-2ax=a^2.$$
$$\therefore x=-\frac{a}{2}.$$

(2) $(x-a)(x-b)-(x-b)(x-c)=2(x-a)(a-c).$

$$(x^2-ax-bx+ab)-(x^2-bx-cx+bc)=2(ax-cx-a^2+ac),$$
$$x^2-ax-bx+ab-x^2+bx+cx-bc=2ax-2cx-2a^2+2ac.$$
$$\text{That is, } -3ax+3cx=-2a^2+2ac-ab+bc,$$
$$-3(a-c)x=-2a(a-c)-b(a-c),$$
$$-3x=-2a-b.$$
$$\therefore x=\frac{2a+b}{3}.$$

Exercise 59.

Solve the equations:

1. $ax+bc=bx+ac.$
2. $2a-cx=3c-5bx.$
3. $a^2x+bx-c=b^2x+cx-d.$
4. $-ac^2+b^2c+abcx=abc+cmx-ac^2x+b^2c-mc.$
5. $(a+x+b)(a+b-x)=(a+x)(b-x)-ab.$
6. $(a^2+x)^2=x^2+4a^2+a^4.$
7. $(a^2-x)(a^2+x)=a^4+2ax-x^2.$
8. $\dfrac{ax-b}{c}+a=\dfrac{x+ac}{c}.$
9. $\dfrac{a(b^2x+x^3)}{bx}=acx+\dfrac{ax^2}{b}.$
10. $ax-\dfrac{3a-bx}{2}=\dfrac{1}{2}.$
11. $6a-\dfrac{4ax-2b}{3}=x.$
12. $\dfrac{x^2-a}{bx}-\dfrac{a-x}{b}=\dfrac{2x}{b}-\dfrac{a}{x}.$
13. $\dfrac{3}{c}-\dfrac{ab-x^2}{bx}=\dfrac{4x-ac}{cx}.$
14. $am-b-\dfrac{ax}{b}+\dfrac{x}{m}=0.$

15. $\dfrac{3ax-2b}{3b}-\dfrac{ax-a}{2b}=\dfrac{ax}{b}-\dfrac{2}{3}.$

16. $\dfrac{ab+x}{b^2}-\dfrac{b^2-x}{a^2b}=\dfrac{x-b}{a^2}-\dfrac{ab-x}{b^2}.$

17. $ax-\dfrac{bx+1}{x}=\dfrac{a(x^2-1)}{x}.$

18. $\dfrac{ax^2}{b-cx}+a+\dfrac{ax}{c}=0.$

19. $\dfrac{ab}{x}=bc+d+\dfrac{1}{x}.$

20. $\dfrac{a(d^2+x^2)}{dx}=ac+\dfrac{ax}{d}.$

Exercise 60.

Solve the equations:

1. $\dfrac{x-3}{4(x-1)}=\dfrac{x-5}{6(x-1)}+\dfrac{1}{9}.$

2. $x+\dfrac{x}{x-1}=\dfrac{(x-2)(x+4)}{x+1}.$

3. $\dfrac{7}{x-1}=\dfrac{6x+1}{x+1}-\dfrac{3(1+2x^2)}{x^2-1}.$

4. $\dfrac{1}{2(x-3)}-\dfrac{1}{3(x-2)}=\dfrac{x-1}{(x-2)(x-3)}.$

5. $1-\dfrac{2(2x+3)}{9(7-x)}=\dfrac{6}{7-x}-\dfrac{5x+1}{4(7-x)}.$

6. $\dfrac{17}{x+3}-4=\dfrac{5(21+2x)}{3x+9}-10.$

7. $\dfrac{x-7}{x+7}=\dfrac{2x-15}{2x-6}-\dfrac{1}{2(x+7)}.$

8. $\dfrac{x+4}{3x+5}+1\tfrac{1}{6}=\dfrac{3x+8}{2x+3}.$

9. $\dfrac{132x+1}{3x+1}+\dfrac{8x+5}{x-1}=52.$

10. $\dfrac{2}{2x-3}+\dfrac{1}{x-2}=\dfrac{6}{3x+2}.$

11. $\dfrac{3x-1}{2x-1}-\dfrac{4x-2}{3x-2}=\dfrac{1}{6}.$

12. $\dfrac{3}{x-1}-\dfrac{x+1}{x-1}=\dfrac{x^2}{1-x^2}.$

13. $\dfrac{x-4}{x-5}-\dfrac{x-5}{x-6}=\dfrac{x-7}{x-8}-\dfrac{x-8}{x-9}.$

14. $(x-a)(x-b)=(x-a-b)^2.$

15. $(a-b)(x-c)-(b-c)(x-a)-(c-a)(x-b)=0.$

16. $\dfrac{x^2-x+1}{x-1}+\dfrac{x^2+x+1}{x+1}=2x.$

17. $\dfrac{4}{x+2}+\dfrac{7}{x+3}=\dfrac{37}{x^2+5x+6}.$

18. $(x+1)^2=x[6-(1-x)]-2.$

19. $\dfrac{25-\frac{1}{3}x}{x+1}+\dfrac{16x+4\frac{1}{5}}{3x+2}=\dfrac{23}{x+1}+5.$

20. $\dfrac{3abc}{a+b}+\dfrac{a^2b^2}{(a+b)^3}+\dfrac{(2a+b)b^2x}{a(a+b)^2}=3cx+\dfrac{bx}{a}.$

21. $\dfrac{4}{x-8}+\dfrac{3}{2x-16}-\dfrac{29}{24}=\dfrac{2}{3x-24}.$

22. $5-x\left(\dfrac{7}{2}-\dfrac{2}{x}\right)=\dfrac{x}{2}-\dfrac{3x-(4-5x)}{4}.$

23. $\dfrac{1}{5}-\dfrac{3}{x-1}=\dfrac{2+\dfrac{x+4}{1-x}}{3}.$

24. $\dfrac{x-\frac{3}{2}}{\frac{3}{2}(x-1)}+\dfrac{x-\frac{5}{2}}{\frac{5}{2}(x+1)}=1+\dfrac{1}{15\left(1-\dfrac{1}{x^2}\right)}.$

154. Problems involving Fractional Equations.

Ex. The sum of the third and fourth parts of a certain number exceeds 3 times the difference of the fifth and sixth parts by 29. Find the number.

Let $x =$ the number.

Then $\frac{x}{3} + \frac{x}{4} =$ the sum of its third and fourth parts,

$\frac{x}{5} - \frac{x}{6} =$ the difference of its fifth and sixth parts,

$3\left(\frac{x}{5} - \frac{x}{6}\right) = 3$ times the difference of its fifth and sixth parts,

$\frac{x}{3} + \frac{x}{4} - 3\left(\frac{x}{5} - \frac{x}{6}\right) =$ the given excess.

But $29 =$ the given excess.

$$\therefore \frac{x}{3} + \frac{x}{4} - 3\left(\frac{x}{5} - \frac{x}{6}\right) = 29.$$

Multiply by 60, the L. C. D. of the fractions.

$$20x + 15x - 36x + 30x = 60 \times 29.$$

Combining, $29x = 60 \times 29.$

$$\therefore x = 60.$$

Exercise 61.

1. Find the number whose third and fourth parts together make 14.
2. Find the number whose third part exceeds its fourth part by 14.
3. The half, fourth, and fifth of a certain number are together equal to 76; find the number.
4. Find the number whose double exceeds its half by 12.
5. Divide 60 into two such parts that a seventh of one part may be equal to an eighth of the other.
6. Divide 50 into two such parts that a fourth of one part increased by five-sixths of the other part may be equal to 40.

7. Divide 100 into two such parts that a fourth of one part diminished by a third of the other part may be equal to 11.

8. The sum of the fourth, fifth, and sixth parts of a certain number exceeds the half of the number by 112. What is the number?

9. The sum of two numbers is 5760, and their difference is equal to one-third of the greater. What are the numbers?

10. Divide 45 into two such parts that the first part divided by 2 shall be equal to the second part multiplied by 2.

11. Find a number such that the sum of its fifth and its seventh parts shall exceed the difference of its fourth and its seventh parts by 99.

12. In a mixture of wine and water, the wine was 25 gallons more than half of the mixture, and the water 5 gallons less than one-third of the mixture. How many gallons were there of each?

13. In a certain weight of gunpowder the saltpetre was 6 pounds more than half of the weight, the sulphur 5 pounds less than the third, and the charcoal 3 pounds less than the fourth of the weight. How many pounds were there of each?

14. Divide 46 into two parts such that if one part be divided by 7, and the other by 3, the sum of the quotients shall be 10.

15. A house and garden cost $850, and five times the price of the house is equal to twelve times the price of the garden. What is the price of each?

16. A man leaves the half of his property to his wife, a sixth to each of his two children, a twelfth to his brother, and the remainder, amounting to \$600, to his sister. What was the amount of his property?

17. The sum of two numbers is a and their difference is b; find the numbers.

18. Find two numbers of which the sum is 70, such that the first divided by the second gives 2 as a quotient and 1 as a remainder.

HINT. $$\frac{\text{Dividend} - \text{Remainder}}{\text{Divisor}} = \text{Quotient.}$$

19. Find two numbers of which the difference is 25, such that the second divided by the first gives 4 as a quotient and 4 as a remainder.

20. Divide the number 208 into two parts such that the sum of the fourth of the greater and the third of the smaller is less by 4 than four times the difference of the two parts.

21. Find four consecutive numbers whose sum is 82.

NOTE. If x represent a person's age at the present time, his age a years ago will be represented by $x - a$, and a years hence by $x + a$.

Ex. In eight years a boy will be three times as old as he was eight years ago. How old is he?

Let $x =$ the number of years of his age.
Then $x - 8 =$ the number of years of his age eight years ago,
and $x + 8 =$ the number of years of his age eight years hence.

Since his age 8 years hence will be three times his age 8 years ago, we have

$$\begin{aligned} x + 8 &= 3(x - 8), \\ x + 8 &= 3x - 24, \\ x - 3x &= -24 - 8, \\ -2x &= -32, \\ x &= 16. \end{aligned}$$

22. A is 72 years old, and B's age is two-thirds of A's. How long is it since A was five times as old as B?

23. A mother is 70 years old, her daughter is half that age. How long is it since the mother was three and one-third times as old as the daughter?

24. A father is three times as old as the son; four years ago the father was four times as old as the son then was. What is the age of each?

25. A is twice as old as B, and seven years ago their united ages amounted to as many years as now represent the age of A. Find the ages of A and B.

26. The sum of the ages of a father and son is half what it will be in 25 years; the difference is one-third what the sum will be in 20 years. What is the age of each?

NOTE. If A can do a piece of work in x days, the *part* of the work that he can do in *one* day will be represented by $\frac{1}{x}$. Thus, if he can do the work in 5 days, in 1 day he can do $\frac{1}{5}$ of the work.

Ex. A can do a piece of work in 5 days, and B can do it in 4 days. How long will it take A and B together?

Let $x =$ the number of days it will take A and B together.
Then $\frac{1}{x} =$ the part they can do in one day.
Now, $\frac{1}{5} =$ the part A can do in one day,
and $\frac{1}{4} =$ the part B can do in one day.
$\therefore \frac{1}{5} + \frac{1}{4} =$ the part A and B can do in one day.

$$\therefore \tfrac{1}{5} + \tfrac{1}{4} = \frac{1}{x},$$
$$4x + 5x = 20,$$
$$9x = 20,$$
$$x = 2\tfrac{2}{9}.$$

Therefore they will do the work in $2\frac{2}{9}$ days.

27. A can do a piece of work in 5 days, B in 6 days, and C in $7\frac{1}{2}$ days; in what time will they do it, all working together?

28. A can do a piece of work in $2\frac{1}{2}$ days, B in $3\frac{1}{3}$ days, and C in $3\frac{3}{4}$ days; in what time will they do it, all working together?

29. Two men who can separately do a piece of work in 15 days and 16 days, can, with the help of another, do it in 6 days. How long would it take the third man to do it alone?

30. A can do half as much work as B, B can do half as much as C, and together they can complete a piece of work in 24 days. In what time can each alone complete the work?

31. A does $\frac{5}{9}$ of a piece of work in 10 days, when B comes to help him, and they finish the work in 3 days more. How long would it have taken B alone to do the whole work?

32. A and B together can reap a field in 12 hours, A and C in 16 hours, and A by himself in 20 hours. In what time can B and C together reap it? In what time can A, B, and C together reap it?

33. A and B together can do a piece of work in 12 days, A and C in 15 days, B and C in 20 days. In what time can they do it, all working together?

NOTE. If a pipe can fill a vessel in x hours, the part of the vessel filled by it in one hour will be represented by $\frac{1}{x}$. Thus, if a pipe will fill a vessel in 3 hours, in 1 hour it will fill $\frac{1}{3}$ of the vessel.

34. A tank can be filled by two pipes in 24 minutes and 30 minutes respectively, and emptied by a third in 20 minutes. In what time will it be filled if all three are running together?

35. A tank can be filled in 15 minutes by two pipes, A and B, running together. After A has been run-

ning by itself for 5 minutes, B is also turned on, and the tank is filled in 13 minutes more. In what time may it be filled by each pipe separately?

36. A cistern could be filled by two pipes in 6 hours and 8 hours respectively, and could be emptied by a third in 12 hours. In what time would the cistern be filled if the pipes were all running together?

37. A tank can be filled by three pipes in 1 hour and 20 minutes, 3 hours and 20 minutes, and 5 hours, respectively. In what time will the tank be filled when all three pipes are running together?

38. If three pipes can fill a cistern in a, b, and c minutes, respectively, in what time will it be filled by all three running together?

39. The capacity of a cistern is $755\frac{1}{4}$ gallons. The cistern has three pipes, of which the first lets in 12 gallons in $3\frac{1}{4}$ minutes, the second $15\frac{1}{3}$ gallons in $2\frac{1}{2}$ minutes, the third 17 gallons in 3 minutes. In what time will the cistern be filled by the three pipes running together?

NOTE. In questions involving distance, time, and rate,

$$\frac{\text{Distance}}{\text{Rate}} = \text{Time}.$$

Thus, if a man travels 40 miles at the rate of 4 miles an hour,

$$\frac{40}{4} = \text{number of hours required}.$$

Ex. A courier who goes at the rate of $31\frac{1}{2}$ miles in 5 hours, is followed, after 8 hours, by another, who goes at the rate of $22\frac{1}{2}$ miles in 3 hours. In how many hours will the second overtake the first?

Since the first goes $31\frac{1}{2}$ miles in 5 hours, his rate per hour is $6\frac{3}{10}$ miles.

Since the second goes $22\frac{1}{2}$ miles in 3 hours, his rate per hour is $7\frac{1}{2}$ miles.

Let $\quad x =$ the number of hours the first is travelling.

Then $\quad x - 8 =$ the number of hours the second is travelling.

Then $\quad 6\frac{3}{10}x =$ the number of miles the first travels;

$(x-8)\,7\frac{1}{2} =$ the number of miles the second travels.

They both travel the same distance.

$$\therefore\ 6\tfrac{3}{10}x = (x-8)\,7\tfrac{1}{2}.$$

The solution of which gives 42 hours.

40. A sets out and travels at the rate of 7 miles in 5 hours. Eight hours afterwards, B sets out from the same place and travels in the same direction, at the rate of 5 miles in 3 hours. In how many hours will B overtake A?

41. A person walks to the top of a mountain at the rate of $2\frac{1}{3}$ miles an hour, and down the same way at the rate of $3\frac{1}{2}$ miles an hour, and is out 5 hours. How far is it to the top of the mountain?

42. A person has a hours at his disposal. How far may he ride in a coach which travels b miles an hour, so as to return home in time, walking back at the rate of c miles an hour?

43. The distance between London and Edinburgh is 360 miles. One traveller starts from Edinburgh and travels at the rate of 10 miles an hour; another starts at the same time from London, and travels at the rate of 8 miles an hour. How far from London will they meet?

44. Two persons set out from the same place in opposite directions. The rate of one of them per hour is a mile less than double that of the other, and in 4 hours they are 32 miles apart. Determine their rates.

45. In going a certain distance, a train travelling 35 miles an hour takes 2 hours less than one travelling 25 miles an hour. Determine the distance.

NOTE. In problems relating to clocks, it is to be noticed that the minute-hand moves *twelve times* as fast as the hour-hand.

Ex. Find the time between 2 and 3 o'clock when the hands of a clock are together.

At 2 o'clock the hour-hand is 10 minute-spaces ahead of the minute-hand.

Let $x =$ the number of spaces the minute-hand moves over.

Then $x - 10 =$ the number of spaces the hour-hand moves over.

Now, as the minute-hand moves 12 times as fast as the hour-hand,

$12(x-10) =$ the number of spaces the minute-hand moves over.

$$\therefore\ x = 12(x-10),$$

and

$$11x = 120.$$

$$\therefore\ x = 10\tfrac{10}{11}.$$

Therefore the time is $10\frac{10}{11}$ minutes past 2 o'clock.

46. At what time are the hands of a watch together:

I. Between 3 and 4?
II. Between 6 and 7?
III. Between 9 and 10?

47. At what time are the hands of a watch at right angles:

I. Between 3 and 4?
II. Between 4 and 5?
III. Between 7 and 8?

48. At what time are the hands of a watch opposite to each other:

I. Between 1 and 2?
II. Between 4 and 5?
III. Between 8 and 9?

49. It is between 2 and 3 o'clock; but a person looking at his watch and mistaking the hour-hand for the minute-hand, fancies that the time of day is 55 minutes earlier than it really is. What is the true time?

NOTE. If a represents the number of feet in the length of a step or leap, and x the number of steps or leaps taken, then ax will represent the number of feet in the distance made.

Ex. A hare takes 4 leaps to a greyhound's 3; but 2 of the greyhound's leaps are equivalent to 3 of the hare's. The hare has a start of 50 leaps. How many leaps must the greyhound take to catch the hare?

Let $3x =$ the number of leaps taken by the greyhound.
Then $4x =$ the number of leaps of the hare in the same time.
Also, let a denote the number of feet in one leap of the hare.
Then $\frac{3a}{2}$ will denote the number of feet in one leap of the greyhound.

That is, $3x \times \frac{3a}{2} =$ the whole distance,

and $(50 + 4x)a =$ the whole distance.

$$\therefore \frac{9ax}{2} = (50 + 4x)a.$$

Divide by a,

$$\frac{9x}{2} = 50 + 4x,$$

$$9x = 100 + 8x,$$

$$x = 100.$$

$$\therefore 3x = 300.$$

Thus the greyhound must take 300 leaps.

50. A hare takes 6 leaps to a dog's 5, and 7 of the dog's leaps are equivalent to 9 of the hare's. The hare has a start of 50 of her own leaps. How many leaps will the hare take before she is caught?

51. A greyhound makes 4 leaps while a hare makes 5; but 3 of the greyhound's leaps are equivalent to 4 of the hare's. The hare has a start of 60 of the greyhound's leaps. How many leaps does each take before the hare is caught?

52. A greyhound makes two leaps while a hare makes 3; but 1 leap of the greyhound is equivalent to 2 of the hare's. The hare has a start of 80 of her own leaps. How many leaps will the hare take before she is caught?

NOTE. If the number of units in the breadth and length of a rectangle is represented by x and $x + a$, respectively, then $x(x + a)$ will represent the number of units of area in the rectangle, the unit of area having the same name as the linear unit in which the sides of the rectangle are expressed.

53. A rectangle whose length is 5 feet more than its breadth would have its area increased by 22 square feet if its length and breadth were each made a foot more. Find its dimensions.

54. A rectangle has its length and breadth respectively 5 feet longer and 3 feet shorter than the side of the equivalent square. Find its area.

55. The length of a rectangle is an inch less than double its breadth; and when a strip 3 inches wide is cut off all round, the area is diminished by 210 inches. Find the size of the rectangle at first.

56. The length of a floor exceeds the breadth by 4 feet; if each dimension were increased by 1 foot, the area of the room would be increased by 27 square feet. Find its dimensions.

NOTE. If b pounds of metal lose a pounds when weighed in water, 1 pound will lose $\frac{1}{b}$ of a pounds, or $\frac{a}{b}$ of a pound.

57. A mass of tin and lead weighing 180 pounds loses 21 pounds when weighed in water; and it is known that 37 pounds of tin lose 5 pounds, and 23 pounds of lead lose 2 pounds, when weighed in water. How many pounds of tin and of lead in the mass?

58. If 19 pounds of gold lose 1 pound, and 10 pounds of silver lose 1 pound, when weighed in water, find the amount of each in a mass of gold and silver weighing 106 pounds in air and 99 pounds in water.

59. Fifteen sovereigns should weigh 77 pennyweights; but a parcel of light sovereigns, having been weighed and counted, was found to contain 9 more than was supposed from the weight; and it appeared that 21 of these coins weighed the same as 20 true sovereigns. How many were there in all?

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60. There are two silver cups, and one cover for both. The first weighs 12 ounces, and with the cover weighs twice as much as the other without it; but the second with the cover weighs one-third more than the first without it. Find the weight of the cover.

61. A man wishes to inclose a circular piece of ground with palisades, and finds that if he sets them a foot apart he will have too few by 150; but if he sets them a yard apart he will have too many by 70. What is the circuit of the piece of ground?

62. A horse was sold at a loss for \$200; but if it had been sold for \$250, the gain would have been three-fourths of the loss when sold for \$200. Find the value of the horse.

63. A and B shoot by turns at a target. A puts 7 bullets out of 12, and B 9 out of 12, into the centre. Between them they put in 32 bullets. How many shots did each fire?

64. A boy buys a number of apples at the rate of 5 for 2 pence. He sells half of them at 2 a penny and the rest at 3 a penny, and clears a penny by the transaction. How many does he buy?

65. A person bought a piece of land for $6750, of which he kept $\frac{4}{9}$ for himself. At the cost of $250 he made a road which took $\frac{1}{10}$ of the remainder, and then sold the rest at $12\frac{1}{2}$ cents a square yard more than double the price it cost him, thus clearing his outlay and $500 besides. How much land did he buy, and what was the cost-price per yard?

66. A boy who runs at the rate of 12 yards per second starts 20 yards behind another whose rate is $10\frac{1}{2}$ yards per second. How soon will the first boy be 10 yards ahead of the second?

67. A merchant adds yearly to his capital one-third of it, but takes from it, at the end of each year, $5000 for expenses. At the end of the third year, after deducting the last $5000, he has twice his original capital. How much had he at first?

68. A shepherd lost a number of sheep equal to one-fourth of his flock and one-fourth of a sheep; then, he lost a number equal to one-third of what he had left and one-third of a sheep; finally, he lost a number equal to one-half of what now remained and one-half a sheep, after which he had but 25 sheep left. How many had he at first?

69. A trader maintained himself for three years at an expense of $250 a year; and each year increased that part of his stock which was not so expended by one-third of it. At the end of the third year his original stock was doubled. What was his original stock?

70. A cask contains 12 gallons of wine and 18 gallons of water; another cask contains 9 gallons of wine and 3 gallons of water. How many gallons must be drawn from each cask to produce a mixture containing 7 gallons of wine and 7 gallons of water?

71. The members of a club subscribe each as many dollars as there are members. If there had been 12 more members, the subscription from each would have been $10 less, to amount to the same sum. How many members were there?

72. A number of troops being formed into a solid square, it was found there were 60 men over; but when formed in a column with 5 men more in front than before, and 3 men less in depth, there was lacking one man to complete it. Find the number of troops.

73. An officer can form the men of his regiment into a hollow square twelve deep. The number of men in the regiment is 1296. Find the number of men in the front of the hollow square.

74. A person starts from P and walks towards Q at the rate of 3 miles an hour; 20 minutes later another person starts from Q and walks towards P at the rate of 4 miles an hour. The distance from P to Q is 20 miles. How far from P will they meet?

75. A person engaged to work a days on these conditions: for each day he worked he was to receive b cents, and for each day he was idle he was to forfeit c cents. At the end of a days he received d cents. How many days was he idle?

76. A banker has two kinds of coins: it takes a pieces of the first to make a dollar, and b pieces of the second to make a dollar. A person wishes to obtain c pieces for a dollar. How many pieces of each kind must the banker give him?

CHAPTER XI.

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE.

155. If we have two unknown numbers and but one relation between them, we can find an unlimited number of pairs of values for which the given relation will hold true. Thus, if x and y are unknown, and we have given only the one relation $x + y = 10$, we can *assume* any value for x, and then from the relation $x + y = 10$ find the corresponding value of y. For from $x + y = 10$ we find $y = 10 - x$. If x stands for 1, y stands for 9; if x stands for 2, y stands for 8; if x stands for -2, y stands for 12; and so on without end.

156. We may, however, have two equations that express *different* relations between the two unknowns. Such equations are called **independent equations.** Thus, $x + y = 10$ and $x - y = 2$ are independent equations, for they evidently express *different* relations between x and y.

157. Independent equations involving the *same* unknowns are called **simultaneous equations.**

If we have two unknowns, and have given two independent equations involving them, there is but *one* pair of values which will hold true for both equations. Thus, if in § 156, besides the relation $x + y = 10$, we have also the relation $x - y = 2$, the only pair of values for which both equations will hold true is the pair $x = 6$, $y = 4$.

Observe that in this problem x stands for the same number in *both* equations; so also does y.

158. Simultaneous equations are solved by combining the equations so as to obtain a single equation with one unknown number; this process is called **elimination.**

There are three methods of elimination in general use:

I. By Addition or Subtraction.
II. By Substitution.
III. By Comparison.

159. Elimination by Addition or Subtraction.

(1) Solve:
$$\left.\begin{aligned} 5x - 3y &= 20 \\ 2x + 5y &= 39 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Multiply (1) by 5, and (2) by 3,
$$\begin{aligned} 25x - 15y &= 100 \qquad (3)\\ 6x + 15y &= 117 \qquad (4)\end{aligned}$$

Add (3) and (4), $\quad 31x = 217$
$$\therefore x = 7.$$

Substitute the value of x in (2),
$$14 + 5y = 39.$$
$$\therefore y = 5.$$

In this solution y is eliminated by *addition.*

(2) Solve:
$$\left.\begin{aligned} 6x + 35y &= 177 \\ 8x - 21y &= 33 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Multiply (1) by 4, and (2) by 3,
$$\begin{aligned} 24x + 140y &= 708 \qquad (3)\\ 24x - 63y &= 99 \qquad (4)\end{aligned}$$

Subtract, $\quad 203y = 609$
$$\therefore y = 3.$$

Substitute the value of y in (2),
$$8x - 63 = 33.$$
$$\therefore x = 12.$$

In this solution x is eliminated by *subtraction.*

160. Hence, to eliminate by addition or subtraction, we have the following rule:

Multiply the equations by such numbers as will make the coefficients of one of the unknown numbers equal in the resulting equations.

Add the resulting equations, or subtract one from the other, according as these equal coefficients have unlike or like signs.

NOTE. It is generally best to select the letter to be eliminated which requires the smallest multipliers to make its coefficients equal; and the smallest multiplier for each equation is found by dividing the L. C. M. of the coefficients of this letter by the given coefficient in that equation. Thus, in example (2), the L. C. M. of 6 and 8 (the coefficients of x) is 24, and hence the smallest multipliers of the two equations are 4 and 3 respectively.

Sometimes the solution is simplified by first adding the given equations, or by subtracting one from the other.

(3)

$$x + 49y = 51 \qquad (1)$$

$$49x + y = 99 \qquad (2)$$

Add (1) and (2), $\qquad 50x + 50y = 150 \qquad (3)$

Divide (3) by 50, $\qquad x + y = 3. \qquad (4)$

Subtract (4) from (1), $\qquad 48y = 48.$

$$\therefore y = 1.$$

Subtract (4) from (2), $\qquad 48x = 96.$

$$\therefore x = 2.$$

Exercise 62.

Solve by addition or subtraction:

1. $\left.\begin{aligned} 2x + 3y &= 7 \\ 4x - 5y &= 3 \end{aligned}\right\}$

2. $\left.\begin{aligned} x - 2y &= 4 \\ 2x - y &= 5 \end{aligned}\right\}$

3. $\left.\begin{aligned} 7x + 2y &= 30 \\ y - 3x &= 2 \end{aligned}\right\}$

4. $\left.\begin{aligned} 3x - 5y &= 51 \\ 2x + 7y &= 3 \end{aligned}\right\}$

5. $\left.\begin{aligned} 5x + 4y &= 58 \\ 3x + 7y &= 67 \end{aligned}\right\}$

6. $\left.\begin{aligned} 3x + 2y &= 39 \\ 3y - 2x &= 13 \end{aligned}\right\}$

7. $\left.\begin{aligned}3x - 4y &= -5\\ 4x - 5y &= 1\end{aligned}\right\}$

8. $\left.\begin{aligned}11x + 3y &= 100\\ 4x - 7y &= 4\end{aligned}\right\}$

9. $\left.\begin{aligned}x + 49y &= 693\\ 49x + y &= 357\end{aligned}\right\}$

10. $\left.\begin{aligned}17x + 3y &= 57\\ 16y - 3x &= 23\end{aligned}\right\}$

11. $\left.\begin{aligned}12x + 7y &= 176\\ 3y - 19x &= 3\end{aligned}\right\}$

12. $\left.\begin{aligned}2x - 7y &= 8\\ 4y - 9x &= 19\end{aligned}\right\}$

13. $\left.\begin{aligned}69y - 17x &= 103\\ 14x - 13y &= -41\end{aligned}\right\}$

14. $\left.\begin{aligned}17x + 30y &= 59\\ 19x + 28y &= 77\end{aligned}\right\}$

161. Elimination by Substitution.

(1) Solve: $\left.\begin{aligned}5x + 4y &= 32\\ 4x + 3y &= 25\end{aligned}\right\}$

$$5x + 4y = 32. \qquad (1)$$

$$4x + 3y = 25. \qquad (2)$$

Transpose $4y$ in (1), $5x = 32 - 4y.$ (3)

Divide by coefficient of x, $x = \frac{32 - 4y}{5}.$ (4)

Substitute the value of x in (2),

$$4\left(\frac{32 - 4y}{5}\right) + 3y = 25,$$

$$\frac{128 - 16y}{5} + 3y = 25,$$

$$128 - 16y + 15y = 125,$$

$$-y = -3.$$

$$\therefore y = 3.$$

Substitute the value of y in (2),

$$4x + 9 = 25.$$

$$\therefore x = 4.$$

Hence, to eliminate by substitution,

From one of the equations obtain the value of one of the unknown numbers in terms of the other.

Substitute for this unknown number its value in the other equation, and reduce the resulting equation.

Exercise 63.

Solve by substitution:

1. $\left.\begin{aligned} 3x - 4y &= 2 \\ 7x - 9y &= 7 \end{aligned}\right\}$

2. $\left.\begin{aligned} 7x - 5y &= 24 \\ 4x - 3y &= 11 \end{aligned}\right\}$

3. $\left.\begin{aligned} 3x + 2y &= 32 \\ 20x - 3y &= 1 \end{aligned}\right\}$

4. $\left.\begin{aligned} 11x - 7y &= 37 \\ 8x + 9y &= 41 \end{aligned}\right\}$

5. $\left.\begin{aligned} 7x + 5y &= 60 \\ 13x - 11y &= 10 \end{aligned}\right\}$

6. $\left.\begin{aligned} 6x - 7y &= 42 \\ 7x - 6y &= 75 \end{aligned}\right\}$

7. $\left.\begin{aligned} 10x + 9y &= 290 \\ 12x - 11y &= 130 \end{aligned}\right\}$

8. $\left.\begin{aligned} 3x - 4y &= 18 \\ 3x + 2y &= 0 \end{aligned}\right\}$

9. $\left.\begin{aligned} 9x - 5y &= 52 \\ 8y - 3x &= 8 \end{aligned}\right\}$

10. $\left.\begin{aligned} 5x - 3y &= 4 \\ 12y - 7x &= 10 \end{aligned}\right\}$

11. $\left.\begin{aligned} 9y - 7x &= 13 \\ 15x - 7y &= 9 \end{aligned}\right\}$

12. $\left.\begin{aligned} 5x - 2y &= 51 \\ 19x - 3y &= 180 \end{aligned}\right\}$

13. $\left.\begin{aligned} 4x + 9y &= 106 \\ 8x + 17y &= 198 \end{aligned}\right\}$

14. $\left.\begin{aligned} 8x + 3y &= 3 \\ 12x + 9y &= 3 \end{aligned}\right\}$

162. Elimination by Comparison.

Solve: $\left.\begin{aligned} 2x - 5y &= 66 \\ 3x + 2y &= 23 \end{aligned}\right\}$

$$2x - 5y = 66. \quad (1)$$
$$3x + 2y = 23. \quad (2)$$

Transpose $5y$ in (1), and $2y$ in (2),

$$2x = 66 + 5y, \quad (3)$$
$$3x = 23 - 2y. \quad (4)$$

Divide (3) by 2, $$x = \frac{66 + 5y}{2}. \quad (5)$$

Divide (4) by 3, $$x = \frac{23 - 2y}{3}. \quad (6)$$

Equate the values of x, $$\frac{66 + 5y}{2} = \frac{23 - 2y}{3}. \quad (7)$$

Reduce (7), $198 + 15y = 46 - 4y,$

$$19y = -152.$$

$$\therefore y = -8.$$

Substitute the value of y in (1),

$$2x + 40 = 66.$$

$$\therefore x = 13.$$

163. Hence, to eliminate by comparison,

From each equation obtain the value of one of the unknown numbers in terms of the other.

Form an equation from these equal values and reduce the equation.

Exercise 64.

Solve by comparison:

1. $\left.\begin{aligned} x + 15y &= 53 \\ 3x + y &= 27 \end{aligned}\right\}$

2. $\left.\begin{aligned} 4x + 9y &= 51 \\ 8x - 13y &= 9 \end{aligned}\right\}$

3. $\left.\begin{aligned} 4x + 3y &= 48 \\ 5y - 3x &= 22 \end{aligned}\right\}$

4. $\left.\begin{aligned} 2x + 3y &= 43 \\ 10x - y &= 7 \end{aligned}\right\}$

5. $\left.\begin{aligned} 5x - 7y &= 33 \\ 11x + 12y &= 100 \end{aligned}\right\}$

6. $\left.\begin{aligned} 5x + 7y &= 43 \\ 11x + 9y &= 69 \end{aligned}\right\}$

7. $\left.\begin{aligned} 8x - 21y &= 33 \\ 6x + 35y &= 177 \end{aligned}\right\}$

8. $\left.\begin{aligned} 3y - 7x &= 4 \\ 2y + 5x &= 22 \end{aligned}\right\}$

9. $\left.\begin{aligned} 21y + 20x &= 165 \\ 77y - 30x &= 295 \end{aligned}\right\}$

10. $\left.\begin{aligned} 11x - 10y &= 14 \\ 5x + 7y &= 41 \end{aligned}\right\}$

11. $\left.\begin{aligned} 7y - 3x &= 139 \\ 2x + 5y &= 91 \end{aligned}\right\}$

12. $\left.\begin{aligned} 17x + 12y &= 59 \\ 19x - 4y &= 153 \end{aligned}\right\}$

13. $\left.\begin{aligned} 24x + 7y &= 27 \\ 8x - 33y &= 115 \end{aligned}\right\}$

14. $\left.\begin{aligned} x &= 3y - 19 \\ y &= 3x - 23 \end{aligned}\right\}$

164. Each equation must be simplified, if necessary, before the elimination.

Solve:
$$\left.\begin{aligned} \tfrac{3}{4}x - \tfrac{1}{2}(y+1) &= 1 \\ \tfrac{1}{3}(x+1) + \tfrac{3}{4}(y-1) &= 9 \end{aligned}\right\}$$

$$\tfrac{3}{4}x - \tfrac{1}{2}(y+1) = 1. \qquad (1)$$
$$\tfrac{1}{3}(x+1) + \tfrac{3}{4}(y-1) = 9. \qquad (2)$$

Multiply (1) by 4, and (2) by 12,
$$3x - 2y - 2 = 4. \qquad (3)$$
$$4x + 4 + 9y - 9 = 108. \qquad (4)$$

From (3), $3x - 2y = 6.$ (5)

From (4), $4x + 9y = 113.$ (6)

Multiply (5) by 4, and (6) by 3,
$$\begin{aligned} 12x - 8y &= 24 \\ 12x + 27y &= 339 \\ \hline 35y &= 315 \\ \therefore y &= 9. \end{aligned}$$

Substitute value of y in (1), $x = 8.$

Exercise 65.

Solve:

1. $\left.\begin{aligned} x(y+7) &= y(x+1) \\ 2x + 20 &= 3y + 1 \end{aligned}\right\}$

2. $\left.\begin{aligned} 2x - \frac{y-3}{5} - 4 &= 0 \\ 3y + \frac{x-2}{3} - 9 &= 0 \end{aligned}\right\}$

3. $\left.\begin{aligned} \frac{2}{x+3} &= \frac{3}{y-2} \\ 5(x+3) &= 3(y-2) + 2 \end{aligned}\right\}$

4. $\left.\begin{aligned} \frac{x-4}{5} - \frac{y+2}{10} &= 0 \\ \frac{x}{6} + \frac{y-2}{4} &= 3 \end{aligned}\right\}$

5. $\left.\begin{aligned} (x+1)(y+2) - (x+2)(y+1) &= -1 \\ 3(x+3) - 4(y+4) &= -8 \end{aligned}\right\}$

6. $\left.\begin{aligned} \frac{x-2}{5} - \frac{10-x}{3} &= \frac{y-10}{4} \\ \frac{2y+4}{3} - \frac{2x+y}{8} &= \frac{x+13}{4} \end{aligned}\right\}$

7. $\left.\begin{aligned}\frac{x+1}{3}-\frac{y+2}{4}&=\frac{2(x-y)}{5}\\ \frac{x-3}{4}-\frac{y-3}{3}&=2y-x\end{aligned}\right\}$

8. $\left.\begin{aligned}\frac{3x-2y}{5}+\frac{5x-3y}{3}&=x+1\\ \frac{2x-3y}{3}+\frac{4x-3y}{2}&=y+1\end{aligned}\right\}$

9. $\left.\begin{aligned}\frac{2x-y+3}{3}-\frac{x-2y+3}{4}&=4\\ \frac{3x-4y+3}{4}+\frac{4x-2y-9}{3}&=4\end{aligned}\right\}$

10. $\left.\begin{aligned}1\tfrac{1}{2}x&=1\tfrac{1}{3}y+4\tfrac{5}{12}\\ 4\tfrac{1}{2}x&=\tfrac{1}{3}y-21\tfrac{7}{12}\end{aligned}\right\}$

11. $\left.\begin{aligned}\frac{13}{x+2y+3}&=-\frac{3}{4x-5y+6}\\ \frac{3}{6x-5y+4}&=\frac{19}{3x+2y+1}\end{aligned}\right\}$

12. $\left.\begin{aligned}\frac{x+y}{y-x}&=\frac{15}{8}\\ 9x-\frac{3y+44}{7}&=100\end{aligned}\right\}$

13. $\left.\begin{aligned}\frac{3x-5y}{2}+3&=\frac{2x+y}{5}\\ 8-\frac{x-2y}{4}&=\frac{x}{2}+\frac{y}{3}\end{aligned}\right\}$

14. $\left.\begin{aligned}\frac{4x-3y-7}{5}&=\frac{3x}{10}-\frac{2y}{15}-\frac{5}{6}\\ \frac{y-1}{3}+\frac{x}{2}-\frac{3y}{20}-1&=\frac{y-x}{15}+\frac{x}{6}+\frac{1}{10}\end{aligned}\right\}$

15. $\left.\begin{aligned}\frac{x-4}{5}&=\frac{y+2}{10}\\ \frac{x}{6}+\frac{y-2}{4}&=3\end{aligned}\right\}$

16. $\left.\begin{aligned}\frac{3x+12y}{11}&=9\\ \frac{1-3x}{7}&=\frac{11-3y}{5}\end{aligned}\right\}$

17. $\left.\begin{aligned}5x-\tfrac{1}{4}(5y+2)&=32\\ 3y+\tfrac{1}{3}(x+2)&=9\end{aligned}\right\}$

18. $\left.\begin{aligned}3x-0.25y&=28\\ 0.12x+0.7y&=2.54\end{aligned}\right\}$

19. $\left.\begin{aligned}7(x-1)&=3(y+8)\\ \frac{4x+2}{9}&=\frac{5y+9}{2}\end{aligned}\right\}$

20. $\left.\begin{aligned}7x+\tfrac{1}{5}(2y+4)&=16\\ 3y-\tfrac{1}{4}(x+2)&=8\end{aligned}\right\}$

21. $$\left.\begin{aligned} \frac{5x-6y}{13}+3x=4y-2 \\ \frac{5x+6y}{6}-\frac{3x-2y}{4}=2y-2 \end{aligned}\right\}$$

22. $$\left.\begin{aligned} \frac{5x-3}{2}-\frac{3x-19}{2}=4-\frac{3y-x}{3} \\ \frac{2x+y}{2}-\frac{9x-7}{8}=\frac{3(y+3)}{4}-\frac{4x+5y}{16} \end{aligned}\right\}$$

23. $$\left.\begin{aligned} 3y+11=\frac{4x^2-y(x+3y)}{x-y+4}+31-4x \\ (x+7)(y-2)+3=2xy-(y-1)(x+1) \end{aligned}\right\}$$

24. $$\left.\begin{aligned} \frac{6x+9}{4}+\frac{3x+5y}{4x-6}=3\tfrac{1}{4}+\frac{3x+4}{2} \\ \frac{8y+7}{10}+\frac{6x-3y}{2y-8}=4+\frac{4y-9}{5} \end{aligned}\right\}$$

25. $$\left.\begin{aligned} x-\frac{2y-x}{23-x}=20-\frac{59-2x}{2} \\ y+\frac{y-3}{x-18}=30-\frac{73-3y}{3} \end{aligned}\right\}$$

165. Literal Simultaneous Equations.

Solve: $$\left.\begin{aligned} ax+by=c \\ a'x+b'y=c' \end{aligned}\right\}$$

NOTE. The letters a', b' are read *a prime, b prime.* In like manner, a'', a''' are read *a second, a third,* and a_1, a_2, a_3 are read *a sub one, a sub two, a sub three.* It is sometimes convenient to represent different numbers that have a common property by the same letter marked by *accents* or *suffixes.* Here a and a' have a common property as coefficients of x.

$$ax+by=c. \qquad (1)$$
$$a'x+b'y=c'. \qquad (2)$$

To find the value of y, multiply (1) by a', and (2) by a,

$$\begin{aligned} aa'x + a'by &= a'c \\ aa'x + ab'y &= ac' \\ \hline a'by - ab'y &= a'c - ac' \\ \therefore y &= \frac{a'c - ac'}{a'b - ab'}. \end{aligned}$$

To find the value of x, multiply (1) by b', and (2) by b, and proceed as in finding the value of y.

Exercise 66.

Solve:

1. $\left.\begin{aligned} x+y&=a \\ x-y&=b \end{aligned}\right\}$

2. $\left.\begin{aligned} ax+by&=c \\ px+qy&=r \end{aligned}\right\}$

3. $\left.\begin{aligned} mx+ny&=a \\ px+qy&=b \end{aligned}\right\}$

4. $\left.\begin{aligned} ax+by&=e \\ ax+cy&=d \end{aligned}\right\}$

5. $\left.\begin{aligned} mx-ny&=r \\ m'x+n'y&=r' \end{aligned}\right\}$

6. $\left.\begin{aligned} ax+by&=c \\ dx+fy&=c^2 \end{aligned}\right\}$

7. $\left.\begin{aligned} \frac{x}{a}+\frac{y}{b}&=c \\ \frac{x}{b}\div\frac{y}{a}&=-c \end{aligned}\right\}$

8. $\left.\begin{aligned} abx+cdy&=2 \\ ax-cy&=\frac{d-b}{bd} \end{aligned}\right\}$

9. $\left.\begin{aligned} \frac{a}{b+y}&=\frac{b}{3a+x} \\ ax+2by&=d \end{aligned}\right\}$

10. $\left.\begin{aligned} \frac{x}{a+b}-\frac{y}{a-b}&=\frac{1}{a+b} \\ \frac{x}{a+b}+\frac{y}{a-b}&=\frac{1}{a-b} \end{aligned}\right\}$

11. $\left.\begin{aligned} a(a-x)&=b(x+y-a) \\ a(y-b-x)&=b(y-b) \end{aligned}\right\}$

12. $\left.\begin{aligned} \frac{x-y+1}{x-y-1}&=a \\ \frac{x+y+1}{x+y-1}&=b \end{aligned}\right\}$

13. $\left.\begin{aligned} ax&=by+\frac{a^2+b^2}{2} \\ (a-b)x&=(a+b)y \end{aligned}\right\}$

14. $\left.\begin{aligned} ax+by&=c^2 \\ \frac{a}{b+y}-\frac{b}{a+x}&=0 \end{aligned}\right\}$

15. $\left.\begin{aligned} \frac{x}{a+b}+\frac{y}{a-b}&=2a \\ \frac{x-y}{2ab}&=\frac{x+y}{a^2+b^2} \end{aligned}\right\}$

16. $\left.\begin{aligned} bx-bc&=ay-ac \\ x-y&=a-b \end{aligned}\right\}$

17. $$\left.\begin{aligned}&\frac{x-a}{y-b}=c\\&a(x-a)+b(y-b)+abc=0\end{aligned}\right\}$$

18. $$\left.\begin{aligned}&(a+b)x-(a-b)y=4ab\\&(a-b)x+(a+b)y=2a^2-2b^2\end{aligned}\right\}$$

19. $$\left.\begin{aligned}&(x+a)(y+b)-(x-a)(y-b)=2(a-b)^2\\&x-y+2(a-b)=0\end{aligned}\right\}$$

20. $$\left.\begin{aligned}&(a+b)(x+y)-(a-b)(x-y)=a^2\\&(a-b)(x+y)+(a+b)(x-y)=b^2\end{aligned}\right\}$$

166. Fractional simultaneous equations, of which the denominators are simple expressions and contain the unknown numbers, may be solved as follows:

(1) Solve: $$\left.\begin{aligned}&\frac{a}{x}+\frac{b}{y}=m\\&\frac{c}{x}+\frac{d}{y}=n\end{aligned}\right\}$$

$$\frac{a}{x}+\frac{b}{y}=m. \tag{1}$$

$$\frac{c}{x}+\frac{d}{y}=n. \tag{2}$$

Multiply (1) by c, $$\frac{ac}{x}+\frac{bc}{y}=cm. \tag{3}$$

Multiply (2) by a, $$\frac{ac}{x}+\frac{ad}{y}=an. \tag{4}$$

Subtract (4) from (3), $$\frac{bc-ad}{y}=cm-an.$$

Multiply both sides by y, $$bc-ad=(cm-an)y.$$

$$\therefore y=\frac{bc-ad}{cm-an}.$$

Multiply (1) by d, $$\frac{ad}{x}+\frac{bd}{y}=dm. \tag{5}$$

Multiply (2) by b, $$\frac{bc}{x}+\frac{bd}{y}=bn. \tag{6}$$

Subtract (6) from (5), $\qquad \dfrac{ad-bc}{x}=dm-bn.$

Multiply both sides by x, $\qquad ad-bc=(dm-bn)x.$

$$\therefore x=\frac{ad-bc}{dm-bn}.$$

(2) Solve:

$$\left.\begin{aligned}\frac{5}{3x}+\frac{2}{5y}&=7\\ \frac{7}{6x}-\frac{1}{10y}&=3\end{aligned}\right\}$$

We have $\qquad \dfrac{5}{3x}+\dfrac{2}{5y}=7, \qquad (1)$

and $\qquad \dfrac{7}{6x}-\dfrac{1}{10y}=3. \qquad (2)$

Multiply (1) by 15, the L. C. M. of 3 and 5, and (2) by 30,

$$\frac{25}{x}+\frac{6}{y}=105. \qquad (3)$$

$$\frac{35}{x}-\frac{3}{y}=90. \qquad (4)$$

Multiply (4) by 2, and add the result to (3),

$$\frac{95}{x}=285.$$

$$\therefore x=\frac{1}{3}.$$

Substitute the value of x in (1), and we get

$$y=\frac{1}{5}.$$

Exercise 67.

Solve:

1. $\left.\begin{aligned}\frac{1}{x}+\frac{2}{y}&=10\\ \frac{4}{x}+\frac{3}{y}&=20\end{aligned}\right\}$

2. $\left.\begin{aligned}\frac{1}{x}+\frac{2}{y}&=a\\ \frac{3}{x}+\frac{4}{y}&=b\end{aligned}\right\}$

3. $\left.\begin{aligned}\frac{2}{x}-\frac{5}{3y}&=\frac{4}{27}\\ \frac{1}{4x}+\frac{1}{y}&=\frac{11}{72}\end{aligned}\right\}$

4. $\left.\begin{aligned}\frac{1}{x}+\frac{2}{y}&=4\\ \frac{3}{x}-\frac{2}{y}&=4\end{aligned}\right\}$

5. $\left.\begin{aligned}\frac{3}{x}-\frac{4}{y}&=5\\ \frac{4}{x}-\frac{5}{y}&=6\end{aligned}\right\}$

6. $\left.\begin{aligned}\frac{a}{x}+\frac{b}{y}&=\frac{ac}{b}\\ \frac{b}{x}+\frac{a}{y}&=\frac{bc}{a}\end{aligned}\right\}$

7. $\left.\begin{aligned}\frac{2}{ax}+\frac{3}{by}&=5\\ \frac{5}{ax}-\frac{2}{by}&=3\end{aligned}\right\}$ 8. $\left.\begin{aligned}\frac{m}{nx}+\frac{n}{my}&=m+n\\ \frac{n}{x}+\frac{m}{y}&=m^2+n^2\end{aligned}\right\}$ 9. $\left.\begin{aligned}\frac{a}{x}+\frac{b}{y}&=m\\ \frac{b}{x}-\frac{a}{y}&=n\end{aligned}\right\}$

167. If three or more simultaneous equations are given, involving three or more unknown numbers, one of the unknowns must be eliminated between *two or more pairs* of the equations; then a second unknown between the pairs that can be formed of the resulting equations.

NOTE. The pairs chosen to eliminate from must be independent pairs, so that *each of the given equations* shall be used in the process of the eliminations.

Solve:
$$\left.\begin{aligned}2x-3y+4z&=4\\ 3x+5y-7z&=12\\ 5x-y-8z&=5\end{aligned}\right\}\quad\begin{aligned}&(1)\\&(2)\\&(3)\end{aligned}$$

Eliminate z between the equations (1) and (3).

Multiply (1) by 2, $4x-6y+8z=8$ (4)

(3) is $5x-y-8z=5$

Add, $9x-7y=13$ (5)

Eliminate z between the equations (1) and (2).

Multiply (1) by 7, $14x-21y+28z=28$

Multiply (2) by 4, $12x+20y-28z=48$

Add, $26x-y=76$ (6)

We now have two equations (5) and (6) involving two unknowns, x and y.

Multiply (6) by 7, $182x-7y=532$ (7)

(5) is $9x-7y=13$

Subtract, $173x=519$

$$\therefore x=3.$$

Substitute the value of x in (6), $78-y=76$.

$$\therefore y=2.$$

Substitute the values of x and y in (1),

$$6-6+4z=4.$$

$$\therefore z=1.$$

Exercise 68.

Solve:

1. $\left.\begin{array}{l} 5x+3y-6z=4 \\ 3x-y+2z=8 \\ x-2y+2z=2 \end{array}\right\}$

2. $\left.\begin{array}{l} 4x-5y+2z=6 \\ 2x+3y-z=20 \\ 7x-4y+3z=35 \end{array}\right\}$

3. $\left.\begin{array}{l} x+y+z=6 \\ 5x+4y+3z=22 \\ 15x+10y+6z=53 \end{array}\right\}$

4. $\left.\begin{array}{l} 4x-3y+z=9 \\ 9x+y-5z=16 \\ x-4y+3z=2 \end{array}\right\}$

5. $\left.\begin{array}{l} 8x+4y-3z=6 \\ x+3y-z=7 \\ 4x-5y+4z=8 \end{array}\right\}$

6. $\left.\begin{array}{l} 12x+5y-4z=29 \\ 13x-2y+5z=58 \\ 17x-y-z=15 \end{array}\right\}$

7. $\left.\begin{array}{l} y-x+z=-5 \\ z-y-x=-25 \\ x+y+z=35 \end{array}\right\}$

8. $\left.\begin{array}{l} x+y+z=30 \\ 8x+4y+2z=50 \\ 27x+9y+3z=64 \end{array}\right\}$

9. $\left.\begin{array}{l} 15y=24z-10x+41 \\ 15x=12y-16z+10 \\ 18x-(7z-13)=14y \end{array}\right\}$

10. $\left.\begin{array}{l} 3x-y+z=17 \\ 5x+3y-2z=10 \\ 7x+4y-5z=3 \end{array}\right\}$

11. $\left.\begin{array}{l} x+y+z=5 \\ 3x-5y+7z=75 \\ 9x-11z+10=0 \end{array}\right\}$

12. $\left.\begin{array}{l} x+2y+3z=6 \\ 2x+4y+2z=8 \\ 3x+2y+8z=101 \end{array}\right\}$

13. $\left.\begin{array}{l} x-3y-2z=1 \\ 2x-3y+5z=-19 \\ 5x+2y-z=12 \end{array}\right\}$

14. $\left.\begin{array}{l} 3x-2y=5 \\ 4x-3y+2z=11 \\ x-2y-5z=-7 \end{array}\right\}$

15. $\left.\begin{array}{l} x+y=1 \\ y+z=9 \\ x+z=5 \end{array}\right\}$

16. $\left.\begin{array}{l} 2x-3y=3 \\ 3y-4z=7 \\ 4z-5x=2 \end{array}\right\}$

17. $\left.\begin{array}{l} 3x-4y+6z=1 \\ 2x+2y-z=1 \\ 7x-6y+7z=2 \end{array}\right\}$

18. $\left.\begin{array}{l} 7x-3y=30 \\ 9y-5z=34 \\ x+y+z=33 \end{array}\right\}$

19. $\left.\begin{aligned} x+\frac{y}{2}+\frac{z}{3}&=6 \\ y+\frac{z}{2}+\frac{x}{3}&=-1 \\ z+\frac{x}{2}+\frac{y}{3}&=17 \end{aligned}\right\}$

20. $\left.\begin{aligned} \frac{1}{x}+\frac{2}{y}&=5 \\ \frac{3}{y}-\frac{4}{z}&=-6 \\ \frac{3}{z}-\frac{4}{x}&=5 \end{aligned}\right\}$

21. $\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}-\frac{1}{z}&=a \\ \frac{1}{x}-\frac{1}{y}+\frac{1}{z}&=b \\ \frac{1}{y}+\frac{1}{z}-\frac{1}{x}&=c \end{aligned}\right\}$ *

22. $\left.\begin{aligned} bz+cy&=a \\ az+cx&=b \\ ay+bx&=c \end{aligned}\right\}$ †

23. $\left.\begin{aligned} \frac{3}{x}-\frac{4}{5y}+\frac{1}{z}&=7\tfrac{3}{5} \\ \frac{1}{3x}+\frac{1}{2y}+\frac{2}{z}&=10\tfrac{1}{6} \\ \frac{4}{5x}-\frac{1}{2y}+\frac{4}{z}&=16\tfrac{1}{10} \end{aligned}\right\}$

24. $\left.\begin{aligned} \frac{2}{x}-\frac{3}{y}+\frac{4}{z}&=2.9 \\ \frac{5}{x}-\frac{6}{y}-\frac{7}{z}&=-10.4 \\ \frac{9}{y}+\frac{10}{z}-\frac{8}{x}&=14.9 \end{aligned}\right\}$

25. $\left.\begin{aligned} \frac{2}{x}+\frac{1}{y}-\frac{3}{z}&=0 \\ \frac{3}{z}-\frac{2}{y}-2&=0 \\ \frac{1}{x}+\frac{1}{z}-\frac{4}{3}&=0 \end{aligned}\right\}$

26. $\left.\begin{aligned} ax+by+cz&=a \\ ax-by-cz&=b \\ ax+cy+bz&=c \end{aligned}\right\}$

27. $\frac{2x-y}{3}=\frac{3y+2z}{4}=\frac{x-y-z}{5}=4.$

28. $\frac{x-y}{a}=\frac{y-z}{b}=\frac{x+z}{c}=\frac{x-a-b}{a+b+c}.$

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* Subtract from the sum of the three equations each equation separately.

† Multiply the equations by a, b, and c, respectively, and from the sum of the results subtract the double of each equation separately.

CHAPTER XII.

PROBLEMS INVOLVING TWO OR MORE UNKNOWN NUMBERS.

168. It is often necessary in the solution of problems to employ two or more letters to represent the numbers to be found. In all cases the conditions must be sufficient to give just as many equations as there are unknown numbers to be found.

169. If there are *more* equations than unknown numbers, some of them are superfluous or inconsistent; if there are *fewer* equations than unknown numbers, the problem is indeterminate.

(1) If A gives B \$10, B will have three times as much money as A. If B gives A \$10, A will have twice as much money as B. How much has each?

Let $x =$ number of dollars A has,
and $y =$ number of dollars B has.

Then, after A gives B \$10,

$$x - 10 = \text{the number of dollars A has,}$$
$$y + 10 = \text{the number of dollars B has.}$$
$$\therefore\ y + 10 = 3(x - 10). \qquad (1)$$

If B gives A \$10,

$$x + 10 = \text{the number of dollars A has,}$$
$$y - 10 = \text{the number of dollars B has.}$$
$$\therefore\ x + 10 = 2(y - 10). \qquad (2)$$

From the solution of equations (1) and (2), $x = 22$, and $y = 26$.

Therefore A has \$22, and B has \$26.

Exercise 69.

1. The sum of two numbers divided by 2 gives as a quotient 24, and the difference between them divided by 2 gives as a quotient 17. What are the numbers?

2. The number 144 is divided into three numbers. When the first is divided by the second, the quotient is 3 and the remainder 2; and when the third is divided by the sum of the other two numbers, the quotient is 2 and the remainder 6. Find the numbers.

3. Three times the greater of two numbers exceeds twice the less by 10; and twice the greater together with three times the less is 24. Find the numbers.

4. If the smaller of two numbers is divided by the greater, the quotient is 0.21 and the remainder 0.0057; but if the greater is divided by the smaller, the quotient is 4 and the remainder 0.742. What are the numbers?

5. Seven years ago the age of a father was four times that of his son; seven years hence the age of the father will be double that of the son. What are their ages?

6. The sum of the ages of a father and son is half what it will be in 25 years; the difference between their ages is one-third of what the sum will be in 20 years. What are their ages?

7. If B gives A $25, they will have equal sums of money; but if A gives B $22, B's money will be double that of A. How much has each?

8. A farmer sold to one person 30 bushels of wheat and 40 bushels of barley for \$67.50; to another person he sold 50 bushels of wheat and 30 bushels of barley for \$85. What was the price of the wheat and of the barley per bushel?

9. If A gives B \$5, he will then have \$6 less than B; but if he receives \$5 from B, three times his money will be \$20 more than four times B's. How much has each?

10. The cost of 12 horses and 14 cows is \$1900; the cost of 5 horses and 3 cows is \$650. What is the cost of a horse and a cow respectively?

NOTE. A fraction of which the terms are unknown may be represented by $\frac{x}{y}$.

Ex. A certain fraction becomes equal to $\frac{1}{2}$ if 3 is added to its numerator, and equal to $\frac{2}{7}$ if 3 is added to its denominator. Determine the fraction.

Let $\frac{x}{y}$ = the required fraction.

Then $\frac{x+3}{y} = \frac{1}{2}$, and $\frac{x}{y+3} = \frac{2}{7}$.

From the solution of these equations it is found that

$$x = 6;$$
$$y = 18.$$

Therefore the fraction $= \frac{6}{18}$.

11. A certain fraction becomes equal to 2 when 7 is added to its numerator, and equal to 1 when 1 is subtracted from its denominator. Determine the fraction.

12. A certain fraction becomes equal to $\frac{1}{2}$ when 7 is added to its denominator, and equal to 2 when 13 is added to its numerator. Determine the fraction.

13. A certain fraction becomes equal to $\frac{7}{9}$ when the denominator is increased by 4, and equal to $\frac{20}{41}$ when the numerator is diminished by 15. Determine the fraction.

14. A certain fraction becomes equal to $\frac{2}{3}$ if 7 is added to the numerator, and equal to $\frac{3}{8}$ if 7 is subtracted from the denominator. Determine the fraction.

15. Find two fractions with numerators 2 and 5 respectively, the sum of which is $1\frac{1}{2}$; but if their denominators are interchanged the sum of the fractions is 2.

16. A fraction which is equal to $\frac{2}{3}$ is increased to $\frac{8}{11}$ when a certain number is added to both its numerator and denominator, and is diminished to $\frac{5}{9}$ when one more than the same number is subtracted from each. Determine the fraction.

NOTE. A number consisting of *two* digits which are unknown may be represented by $10x + y$, in which x and y represent the digits of the number. Likewise, a number consisting of *three* digits which are unknown may be represented by $100x + 10y + z$, in which x, y, and z represent the digits of the number.

For example, consider any number expressed by three digits, as 364. The expression 364 means $300 + 60 + 4$; or, 100 *times* $3 + 10$ *times* $6 + 4$.

Ex. The sum of the two digits of a number is 8, and if 36 be added to the number, the digits will be interchanged. What is the number?

Let $x =$ the digit in the tens' place,
and $y =$ the digit in the units' place.

Then $10x + y =$ the number.

By the conditions, $x + y = 8$, (1)
and $10x + y + 36 = 10y + x$. (2)

From (2), $9x - 9y = -36$.
Divide by 9, $x - y = -4$. (3)

Add (1) and (3), $2x = 4.$

$\therefore\ x = 2.$

Subtract (3) from (1), $2y = 12.$

$\therefore\ y = 6.$

Hence the number is 26.

17. The sum of the two digits of a number is 10, and if 54 be added to the number the digits will be interchanged. What is the number?

18. The sum of the two digits of a number is 6, and if the number is divided by the sum of the digits the quotient is 4. What is the number?

19. A certain number is expressed by two digits, of which the tens' digit is the greater. If the number is divided by the sum of its digits the quotient is 7; if the digits are interchanged, and the resulting number diminished by 12 is divided by the difference between the two digits, the quotient is 9. What is the number?

20. If a certain number is divided by the sum of its two digits the quotient is 6 and the remainder 3; if the digits are interchanged, and the resulting number is divided by the sum of the digits, the quotient is 4 and the remainder 9. What is the number?

21. If a certain number is divided by the sum of its two digits diminished by 2, the quotient is 5 and the remainder 1; if the digits are interchanged, and the resulting number is divided by the sum of the digits increased by 2, the quotient is 5 and the remainder 8. Find the number.

22. The first of the two digits of a number is, when doubled, 3 more than the second, and the number itself is less by 6 than 5 times the sum of the digits. What is the number?

23. A number is expressed by three digits, of which the first and last are alike. By interchanging the digits in the units' and tens' places the number is increased by 54; but if the digits in the tens' and hundreds' places are interchanged, 9 must be added to four times the resulting number to make it equal to the original number. What is the number?

24. A number is expressed by three digits. The sum of the digits is 21; the sum of the first and second exceeds the third by 3; and if 198 be added to the number, the digits in the units' and hundreds' places will be interchanged. Find the number.

25. A number is expressed by three digits. The sum of the digits is 9; the number is equal to forty-two times the sum of the first and second digits; and the third digit is twice the sum of the other two. Find the number.

26. A certain number, expressed by three digits, is equal to forty-eight times the sum of its digits. If 198 be subtracted from the number, the digits in the units' and hundreds' places will be interchanged; and the sum of the extreme digits is equal to twice the middle digit. Find the number.

NOTE. If a boat moves at the rate of x miles an hour in still water, and if it is on a stream that runs at the rate of y miles an hour, then

$x + y$ represents its rate *down* the stream,
$x - y$ represents its rate *up* the stream.

27. A waterman rows 30 miles and back in 12 hours. He finds that he can row 5 miles with the stream in the same time as 3 against it. Find the time it takes him to row up and down respectively.

28. A crew which can pull at the rate of 12 miles an hour down the stream, finds that it takes twice as long to come up the river as to go down. At what rate does the stream flow?

29. A man sculls down a stream, which runs at the rate of 4 miles an hour, for a certain distance in 1 hour and 40 minutes. In returning it takes him 4 hours and 15 minutes to arrive at a point 3 miles short of his starting-place. Find the distance he pulled down the stream and the rate of his pulling.

30. A person rows down a stream a distance of 20 miles and back again in 10 hours. He finds he can row 2 miles against the stream in the same time he can row 3 miles with it. Find the time of his rowing down and of his rowing up the stream; and also the rate of the stream.

NOTE. When commodities are mixed, the quantity of the mixture is equal to the quantity of the ingredients; the cost of the mixture is equal to the cost of the ingredients.

Ex. A wine-merchant has two kinds of wine which cost 72 cents and 40 cents a quart respectively. How much of each must he take to make a mixture of 50 quarts worth 60 cents a quart?

Let $x =$ required number of quarts worth 72 cents a quart,

and $y =$ required number of quarts worth 40 cents a quart.

Then, $72x =$ cost in cents of the first kind,

$40y =$ cost in cents of the second kind of wine,

and $3000 =$ cost in cents of the mixture.

$$\therefore\ x + y = 50,$$

$$72x + 40y = 3000.$$

From which equations the values of x and y may be found.

31. A grocer mixed tea that cost him 42 cents a pound with tea that cost him 54 cents a pound. He had 30 pounds of the mixture, and by selling it at the rate of 60 cents a pound, he gained as much as 10 pounds of the cheaper tea cost him. How many pounds of each did he put into the mixture?

32. A grocer mixes tea that cost him 90 cents a pound with tea that cost him 28 cents a pound. The cost of the mixture is \$61.20. He sells the mixture at 50 cents a pound, and gains \$3.80. How many pounds of each did he put into the mixture?

33. A farmer has 28 bushels of barley worth 84 cents a bushel. With his barley he wishes to mix rye worth \$1.08 a bushel, and wheat worth \$1.44 a bushel, so that the mixture may be 100 bushels, and be worth \$1.20 a bushel. How many bushels of rye and of wheat must he take?

Ex. A and B together can do a piece of work in 48 days; A and C together can do it in 30 days; B and C together can do it in $26\frac{2}{3}$ days. How long will it take each to do the work?

Let $x =$ the number of days it will take A alone to do the work,
$y =$ the number of days it will take B alone to do the work,
and $z =$ the number of days it will take C alone to do the work.

Then, $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$, respectively, will denote the part each can do in a day,

and $\frac{1}{x}+\frac{1}{y}$ will denote the part A and B together can do in a day;

but $\frac{1}{48}$ will denote the part A and B together can do in a day.

Therefore, $$\frac{1}{x}+\frac{1}{y}=\frac{1}{48}. \qquad (1)$$

Likewise, $$\frac{1}{x}+\frac{1}{z}=\frac{1}{30},\tag{2}$$

and $$\frac{1}{y}+\frac{1}{z}=\frac{1}{26\frac{2}{3}}.\tag{3}$$

The solution of these equations gives 120, 80, and 40 for the values of x, y, and z, respectively.

34. A and B together earn \$40 in 6 days; A and C together earn \$54 in 9 days; B and C together earn \$80 in 15 days. What does each earn a day?

35. A cistern has three pipes, A, B, and C. A and B will fill it in 1 hour and 10 minutes; A and C in 1 hour and 24 minutes; B and C in 2 hours and 20 minutes. How long will it take each to fill it?

36. A warehouse will hold 24 boxes and 20 bales; 6 boxes and 14 bales will fill half of it. How many of each alone will it hold?

37. Two workmen together complete some work in 20 days; but if the first had worked twice as fast, and the second half as fast, they would have finished it in 15 days. How long would it take each alone to do the work?

38. A purse holds 19 crowns and 6 guineas; 4 crowns and 5 guineas fill $\frac{17}{63}$ of it. How many of each alone will it hold?

39. A piece of work can be completed by A, B, and C together in 10 days; by A and B together in 12 days; by B and C, if B work 15 days and C 30 days. How long will it take each alone to do the work?

40. A cistern has three pipes, A, B, and C. A and B will fill it in a minutes; A and C in b minutes; B and C in c minutes. How long will it take each alone to fill it?

NOTE. In considering *the rate of increase or decrease* in quantities, it is usual to take 100 as a *common standard of reference*, so that the increase or decrease is calculated for every 100, and therefore called *per cent.*

The representative of the number resulting after an increase has taken place is 100 + increase per cent; and after a decrease, 100 − decrease per cent.

Interest depends upon the *time* for which the money is lent, as well as upon the *rate per cent* charged; the rate per cent charged being the rate per cent on the principal for *one year*. Hence,

$$\text{Simple interest} = \frac{\text{Principal} \times \text{Rate per cent} \times \text{Time}}{100},$$

where Time means *number of years or fraction of a year.*

$$\text{Amount} = \text{Principal} + \text{Interest.}$$

In questions relating to stocks, 100 is taken as the representative of the *stock*, the *price* represents its market value, and the *per cent* represents the *interest* which the *stock* bears. Thus, if six per cent stocks are quoted at 108, the meaning is, that the price of \$100 of the stock is \$108, and that the interest derived from \$100 of the *stock* will be $\frac{6}{100}$ of \$100; that is, \$6 a year. The rate of interest on the *money invested* will be $\frac{100}{108}$ of 6 per cent.

41. A man has \$10,000 invested. For a part of this sum he receives 5 per cent interest, and for the rest 4 per cent; the income from his 5 per cent investment is \$50 more than from his 4 per cent. How much has he in each investment?

42. A sum of money, at simple interest, amounted in 6 years to \$26,000, and in 10 years to \$30,000. Find the sum and the rate of interest.

43. A sum of money, at simple interest, amounted in 10 months to \$26,250, and in 18 months to \$27,250. Find the sum and the rate of interest.

44. A sum of money, at simple interest, amounted in m years to a dollars, and in n years to b dollars. Find the sum and the rate of interest.

45. A sum of money, at simple interest, amounted in a months to c dollars, and in b months to d dollars. Find the sum and the rate of interest.

46. A person has a certain capital invested at a certain rate per cent. Another person has $1000 more capital, and his capital invested at one per cent better than the first, and receives an income $80 greater. A third person has $1500 more capital, and his capital invested at two per cent better than the first, and receives an income $150 greater. Find the capital of each, and the rate at which it is invested.

47. A person has $12,750 to invest. He can buy three per cent bonds at 81, and five per cents at 120. Find the amount of money he must invest in each in order to have the same income from each investment.

Hint. If x and y represent the number of dollars invested in the three and five per cents respectively, then $\frac{3x}{81}$ and $\frac{5y}{120}$ are the respective incomes from the three and five per cents.

48. A and B each invested $1500 in bonds; A in three per cents and B in four per cents. The bonds were bought at such prices that B received $5 interest more than A. After both classes of bonds rose 10 points, they sold out, and A received $50 more than B. What price was paid for each class of bonds?

Hint. If x and y represent the cost of $1 in three per cents and $1 in four per cents, respectively, then $\frac{1500}{x}$ and $\frac{1500}{y}$ are the face values of the three and four per cents, respectively; and $\frac{3}{100}\times\frac{1500}{x}$ and $\frac{4}{100}\times\frac{1500}{y}$ are the respective incomes.

49. A person invests \$10,000 in three per cent bonds, \$16,500 in three and one-half per cents, and has an income from both investments of \$1056.25. If his investments had been \$2750 more in the three per cents, and less in the three and one-half per cents, his income would have been $62\frac{1}{2}$ cents greater. What price was paid for each class of bonds?

HINT. Let x and y represent the cost of \$1 in three and three and one-half per cent bonds respectively; then $\frac{3}{100} \times \frac{10000}{x}$ and $\frac{3\frac{1}{2}}{100} \times \frac{16500}{y}$ are the respective incomes.

50. The sum of \$2500 was divided into two unequal parts and invested, the smaller part at two per cent more than the larger. The *rate* of interest on the larger sum was afterwards increased by 1, and that on the smaller sum diminished by 1; and thus the *interest* of the whole was increased by one-fourth of its value. If the interest of the larger sum had been so increased, and no change been made in the interest of the smaller sum, the interest of the whole would have been increased one-third of its value. Find the sums invested, and the rate per cent of each.

NOTE. If x represents the number of linear units in the length, and y in the width, of a rectangle, xy represents the number of its units of surface; the surface unit having the same name as the linear unit of its sides.

51. If the sides of a rectangular field were each increased by 2 yards, the area would be increased by 220 square yards; if the length were increased and the breadth were diminished each by 5 yards, the area would be diminished by 185 square yards. What is its area?

52. If a given rectangular floor had been 3 feet longer and 2 feet broader it would have contained 64 square feet more; but if it had been 2 feet longer and 3 feet broader it would have contained 68 square feet more. Find the length and breadth of the floor.

53. In a certain rectangular garden there is a strawberry-bed whose sides are one-third of the lengths of the corresponding sides of the garden. The perimeter of the garden exceeds that of the bed by 200 yards; and if the greater side of the garden be increased by 3, and the other by 5 yards, the garden will be enlarged by 645 square yards. Find the length and breadth of the garden.

NOTE. Care must be taken to express the conditions of a problem in the same principal unit.

Ex. In a mile race A gives B a start of 20 yards and beats him by 30 seconds. At the second trial A gives B a start of 32 seconds and beats him by $9\frac{5}{11}$ yards. Find the rate per hour at which each runs.

Let $x =$ number of yards A runs a second,

and $y =$ number of yards B runs a second.

Since there are 1760 yards in a mile,

$\frac{1760}{x} =$ number of seconds it takes A to run a mile.

$\frac{1740}{y}$ and $\frac{1750\frac{6}{11}}{y} =$ number of seconds B was running in the first and second trials, respectively.

$$\text{Hence, } \frac{1740}{y} - \frac{1760}{x} = 30,$$

$$\text{and } \frac{1750\frac{6}{11}}{y} - \frac{1760}{x} = 32.$$

The solution of these equations gives $x = 5\frac{13}{15}$ and $y = 5\frac{3}{11}$.

That is, A runs $\frac{5\frac{13}{15}}{1760}$, or $\frac{1}{300}$, of a mile in one second; and in one hour, or 3600 seconds, runs 12 miles.

Likewise, B runs $10\frac{95}{121}$ miles in one hour.

54. In a mile race A gives B a start of 100 yards and beats him by 15 seconds. In the second trial A gives B a start of 45 seconds and is beaten by 22 yards. Find the rate of each in miles per hour.

55. In a mile race A gives B a start of 44 yards and beats him by 51 seconds. In the second trial A gives B a start of 1 minute and 15 seconds and is beaten by 88 yards. Find the rate of each in miles per hour.

56. The time which an express train takes to go 120 miles is $\frac{9}{14}$ of the time taken by an accommodation train. The slower train loses as much time in stopping at different stations as it would take to travel 20 miles without stopping; the express train loses only half as much time by stopping as the accommodation train, and travels 15 miles an hour faster. Find the rate of each train in miles per hour.

HINT. If x and y represent the rates of the slower and faster trains respectively, then $\frac{120}{x}$ and $\frac{120}{y}$ represent the number of hours it takes the respective trains to run 120 miles; $\frac{20}{x}$ and $\frac{10}{x}$ represent the number of hours the respective trains lose by stopping.

57. A train moves from P towards Q, and an hour later a second train starts from Q and moves towards P at a rate of 10 miles an hour more than the first train; the trains meet half-way between P and Q. If the train from P had started an hour after the train from Q, its rate must have been increased by 28 miles in order that the trains should meet at the half-way point. Find the distance from P to Q.

HINT. If x denotes the number of hours it takes the first train to go half the distance, and y denotes the rate of the first train, then $x-1$ denotes the number of hours it takes the second train to go half the distance, and $y+10$ denotes the rate of the second train. Hence, xy and $(x-1)(y+10)$ are each equal to half the distance.

58. A passenger train, after travelling an hour, meets with an accident which detains it one-half an hour; after which it proceeds at four-fifths of its usual rate, and arrives an hour and a quarter late. If the accident had happened 30 miles farther on, the train would have been only an hour late. Determine the usual rate of the train.

HINT. If x represents the number of miles the train usually goes an hour, and y the whole distance in miles, $y-x$ is the distance the train has to go after the accident, $\frac{y-x}{x}$ is the number of hours usually required, and $\frac{y-x}{\frac{4}{5}x}$ is the number of hours required after the accident. Hence $\frac{y-x}{\frac{4}{5}x}-\frac{y-x}{x}$ is the *loss in running time.*

Also, since the detention is $\frac{1}{2}$ hour, and the train is $1\frac{1}{4}$ hours late, the *loss in running time* is $\frac{3}{4}$ of an hour.

Therefore, $$\frac{y-x}{\frac{4}{5}x}-\frac{y-x}{x}=\frac{3}{4}.$$

If the accident had happened 30 miles farther on, the distance to be run would have been $y-(x+30)$, and we should have had

$$\frac{y-(x+30)}{\frac{4}{5}x}-\frac{y-(x+30)}{x}=\frac{1}{2}.$$

59. A passenger train after travelling an hour is detained 15 minutes; after which it proceeds at three-fourths of its former rate, and arrives 24 minutes late. If the detention had taken place 5 miles farther on, the train would have been only 21 minutes late. Determine the usual rate of the train.

60. A man bought 10 oxen, 120 sheep, and 46 lambs. The cost of 3 sheep was equal to that of 5 lambs; an ox, a sheep, and a lamb together cost a number of dollars less by 57 than the whole number of animals bought; and the whole sum spent was \$2341.50. Find the price of an ox, a sheep, and a lamb, respectively.

61. A farmer sold 100 head of stock, consisting of horses, oxen, and sheep, so that the whole realized \$11.75 a head; while a horse, an ox, and a sheep were sold for \$110, \$62.50, and \$7.50, respectively. Had he sold one-fourth of the number of oxen that he did, and 25 more sheep, he would have received the same sum. Find the number of horses, oxen, and sheep, respectively, which were sold.

62. A, B, and C together subscribed \$100. If A's subscription had been one-tenth less, and B's one-tenth more, C's must have been increased by \$2 to make up the sum; but if A's had been one-eighth more, and B's one-eighth less, C's subscription would have been \$17.50. What did each subscribe?

63. A gives to B and C as much as each of them has; B gives to A and C as much as each of them then has; and C gives to A and B as much as each of them then has. In the end each of them has \$6. How much had each at first?

64. A pays to B and C as much as each of them has; B pays to A and C one-half as much as each of them then has; and C pays to A and B one-third of what each of them then has. In the end A finds that he has \$1.50, B \$4.16$\frac{2}{3}$, C \$0.58$\frac{1}{3}$. How much had each at first?

170. Discussion of Problems. The *discussion* of a problem consists in making various suppositions as to the relative values of the given numbers, and explaining the results. We will illustrate by an example:

Two couriers, A and B, were travelling along the same road, and in the same direction, from C towards D. A travels at the rate of m miles an hour, and B at the rate

of n miles an hour. At 12 o'clock B was d miles in advance of A. When will the couriers be together?

Suppose they will be together x hours *after* 12. Then A has travelled mx miles, and B has travelled nx miles, and as A has travelled d miles more than B,

$$\therefore\ mx = nx + d,$$

or

$$mx - nx = d.$$

$$\therefore\ x = \frac{d}{m-n}.$$

DISCUSSION OF THE PROBLEM. 1. If m is greater than n, the value of x, namely, $\frac{d}{m-n}$, is positive, and it is evident that A will overtake B *after* 12 o'clock.

2. If m is less than n, then $\frac{d}{m-n}$ will be negative. In this case B travels faster than A, and as he is d miles ahead of A at 12 o'clock it is evident that A cannot overtake B *after* 12 o'clock, but that B passed A *before* 12 o'clock. The supposition, therefore, that the couriers were together *after* 12 o'clock was incorrect, and the *negative* value of x points to an **error in the supposition.**

3. If m equals n, then the value of x, that is, $\frac{d}{m-n}$, assumes the form $\frac{d}{0}$. Now if the couriers were d miles apart at 12 o'clock, and if they had been travelling at the same rates, and continue to travel at the same rates, it is obvious that they never had been together, and that they never will be together, so that the symbol $\frac{d}{0}$ may be regarded as the **symbol of impossibility.**

4. If m equals n and d is 0, then $\frac{d}{m-n}$ becomes $\frac{0}{0}$. Now if the couriers were together at 12 o'clock, and if they had been travelling at the same rates, and continue to travel at the same rates, it is obvious that they had been together all the time, and that they will continue to be together all the time, so that there is an indefinite number of solutions. Therefore, the symbol $\frac{0}{0}$ may be regarded as the **symbol of indetermination.**

CHAPTER XIII.

SIMPLE INDETERMINATE EQUATIONS.

171. If one equation involving two unknown numbers is given, and no other condition is imposed, the number of solutions of the equation is unlimited; for if one of the unknown numbers be assumed to have *any* particular value, a corresponding value of the other may be found.

Such an equation is called an **indeterminate equation.**

Although the number of solutions of an indeterminate equation is unlimited, the values of the unknown numbers are confined to a particular range; this range may be further limited by some restriction, as for example by requiring that the unknown numbers shall be *positive integers*.

172. Every indeterminate equation of the first degree, in which x and y are the unknown numbers, may be made to assume the form

$$ax \pm by = \pm c,$$

where a, b, and c are positive integers and have no common factor.

NOTE. The sign $\pm$ is read *plus or minus*, and the sign $\mp$ is read *minus or plus*.

173. Examples.

(1) Solve $3x + 4y = 22$, in positive integers.

Transpose, $$3x = 22 - 4y.$$

$$\therefore x = 7 - y + \frac{1-y}{3},$$

the quotient being written as a mixed expression.

$$\therefore x + y - 7 = \frac{1-y}{3}.$$

Since the values of x and y are to be integral, $x + y - 7$ will be integral, and hence $\frac{1-y}{3}$ will be integral, though written in the *form* of a fraction.

Let $$\frac{1-y}{3} = m, \text{ an integer.}$$

Then $$1 - y = 3m.$$

$$\therefore y = 1 - 3m.$$

Substitute this value of y in the original equation,

$$3x + 4 - 12m = 22.$$

$$\therefore x = 6 + 4m.$$

The equation $y = 1 - 3m$ shows that m may be 0, or have any negative integral value, but cannot have a positive integral value.

The equation $x = 6 + 4m$ further shows that m may be 0, but cannot have a negative integral value greater than 1.

$$\therefore m \text{ may be } 0 \text{ or } -1;$$

and then $$\left.\begin{array}{l} x = 6 \\ y = 1 \end{array}\right\}, \text{ or } \left.\begin{array}{l} x = 2 \\ y = 4 \end{array}\right\}.$$

(2) Solve $5x - 14y = 11$, in positive integers.

Transpose, $$5x = 11 + 14y,$$

$$x = 2 + 2y + \frac{1+4y}{5}. \quad (1)$$

$$\therefore x - 2y - 2 = \frac{1+4y}{5}.$$

Since x and y are to be integral, $x - 2y - 2$ will be integral, and hence $\frac{1+4y}{5}$ will be integral.

Let $$\frac{1+4y}{5} = m, \text{ an integer.}$$

Then $$y = \frac{5m-1}{4},$$

or $$y = m + \frac{m-1}{4}. \quad (2)$$

Now $\frac{m-1}{4}$ must be integral.

Let $\quad \dfrac{m-1}{4} = n$, an integer.

Then $\quad m = 4n + 1.$

Substituting in (2), $\quad y = 5n + 1.$

Substituting in (1), $\quad x = 14n + 5.$

Obviously x and y will both be positive integers if n have *any* positive integral value.

Hence, $\quad x = 5, \quad 19, \quad 33, \quad 47, \quad \ldots\ldots$

$\quad y = 1, \quad 6, \quad 11, \quad 16, \quad \ldots\ldots$

It will be seen from (1) and (2) that when only positive integers are required, the number of solutions will be *limited* or *unlimited* according as the sign connecting x and y is *positive* or *negative.*

(3) Find the least number that when divided by 14 and 5 will give remainders 1 and 3 respectively.

If N represents the number, then

$$\frac{N-1}{14} = x, \text{ and } \frac{N-3}{5} = y.$$

$$\therefore N = 14x + 1, \text{ and } N = 5y + 3.$$

$$\therefore 14x + 1 = 5y + 3.$$

$$5y = 14x - 2,$$

$$5y = 15x - 2 - x.$$

$$\therefore y = 3x - \frac{2+x}{5}.$$

Let $\quad \dfrac{2+x}{5} = m$, an integer.

$$\therefore x = 5m - 2.$$

$y = \frac{1}{5}(14x - 2)$, from the original equation.

$$\therefore y = 14m - 6.$$

If $m = 1$, $\quad x = 3$, and $y = 8$.

$$\therefore N = 14x + 1 = 5y + 3 = 43.$$

(4) Solve $5x + 6y = 30$, so that x may be a multiple of y, and both x and y positive.

Let $\quad x = my.$

Then $\quad (5m + 6)y = 30.$

$$\therefore y = \frac{30}{5m+6},$$

and $$x = \frac{30m}{5m+6}.$$

If $m = 2$, $\quad x = 3\frac{3}{4},\ y = 1\frac{7}{8}.$

If $m = 3$, $\quad x = 4\frac{2}{7},\ y = 1\frac{3}{7}.$

(5) Solve $14x + 22y = 71$, in positive integers.

$$x = 5 - y + \frac{1-8y}{14}.$$

If we multiply the fraction by 7 and reduce, the result is

$$-4y + \tfrac{1}{2},$$

a form which shows that there can be no *integral* solution.

There can be no integral solution of $ax \pm by = \pm c$ if a and b have a common factor not common also to c; for if d be a factor of a and also of b, but not of c, the equation may be written

$$mdx \pm ndy = \pm c, \text{ or } mx \pm ny = \pm \frac{c}{d};$$

which is impossible, since $\frac{c}{d}$ is a fraction, and $mx \pm ny$ is an integer, if x and y are integers.

Exercise 70.

Solve in positive integers:

1. $2x + 11y = 49.$
2. $7x + 3y = 40.$
3. $5x + 7y = 53.$
4. $x + 10y = 29.$
5. $3x + 8y = 61.$
6. $8x + 5y = 97.$
7. $16x + 7y = 110.$
8. $7x + 10y = 206.$

Solve in least positive integers:

9. $12x - 7y = 1.$
10. $5x - 17y = 23.$
11. $23y - 13x = 3.$
12. $23x - 9y = 929.$
13. $23x - 33y = 43.$
14. $555x - 22y = 73.$

15. How many fractions are there with denominators 12 and 18 whose sum is $\frac{25}{36}$?

16. What is the least number which, when divided by 3 and 5, leaves remainders 2 and 3 respectively?

17. A person counting a basket of eggs, which he knows are between 50 and 60, finds that when he counts them 3 at a time there are 2 over; but when he counts them 5 at a time there are 4 over. How many are there in all?

18. A person bought 40 animals, consisting of pigs, geese, and chickens, for \$40. The pigs cost \$5 apiece, the geese \$1, and the chickens 25 cents each. Find the number he bought of each.

19. Find the least multiple of 7 which, when divided by 2, 3, 4, 5, 6, leaves in each case 1 for a remainder.

20. In how many ways may 100 be divided into two parts, one of which shall be a multiple of 7 and the other of 9?

21. Solve $18x - 5y = 70$ so that y may be a multiple of x, and both positive.

22. Solve $8x + 12y = 23$ so that x and y may be positive, and their sum an integer.

23. Divide 70 into three parts which shall give integral quotients when divided by 6, 7, 8, respectively, and the sum of the quotients shall be 10.

24. Divide 200 into three parts which shall give integral quotients when divided by 5, 7, 11, respectively, and the sum of the quotients shall be 20.

25. A number consisting of three digits, of which the middle one is 4, has the digits in the units' and hundreds' places interchanged by adding 792. Find the number.

26. Some men earning each \$2.50 a day, and some women earning each \$1.75 a day, receive all together for their daily wages \$44.75. Determine the number of men and the number of women.

27. A wishes to pay B a debt of £1 12*s*., but has only half-crowns in his pocket, while B has only 4 penny-pieces. How may they settle the matter most simply?

28. Show that $323x - 527y = 1000$ cannot be satisfied by integral values of x and y.

29. A farmer buys oxen, sheep, and hens. The whole number bought is 100, and the whole price £100. If the oxen cost £5, the sheep £1, and the hens 1*s*. each, how many of each did he buy?

30. A number of lengths 3 feet, 5 feet, and 8 feet are cut; how may 48 of them be taken so as to measure 175 feet all together?

31. A field containing an integral number of acres less than 10 is divided into 8 lots of one size, and 7 of 4 times that size, and has also a road passing through it containing 1300 square yards. Find the size of the lots in square yards.

32. Two wheels are to be made, the circumference of one of which is to be a multiple of the other. What circumferences may be taken so that when the first has gone round three times and the other five, the difference in the lengths of rope coiled on them may be 17 feet?

33. In how many ways can a person pay a sum of £15 in half-crowns, shillings, and sixpences, so that the number of shillings and sixpences together shall be equal to the number of half-crowns?

CHAPTER XIV.

INEQUALITIES.

174. Different expressions containing any given letter will have their values changed when different values are assigned to that letter; one expression may be for some values of the letter greater than the other, and for some values of the letter smaller than the other.

175. Two expressions, however, may be so related that, whatever values may be given to the letter, one of the expressions cannot be greater than the other.

Thus, $2x \ngtr x^2 + 1$, whatever value be given to x.

NOTE. The signs $\nless$ and $\ngtr$ are read *not less than* and *not greater than*, respectively.

176. For finding whether this relation holds between two expressions, the following is a fundamental proposition:

If a and b are unequal, $a^2 + b^2 > 2ab$.

For, $(a - b)^2$ must be positive, whatever the values of a and b.

That is, $$(a - b)^2 > 0,$$

or $$a^2 - 2ab + b^2 > 0.$$

$$\therefore\ a^2 + b^2 > 2ab.$$

177. The principles applied to the solution of equations may be applied to inequalities, except that if each side of an equality have its *sign changed*, the inequality will be *reversed*.

Thus, if $a > b$; then $-a < -b$.

(1) If a and b are positive, show that $a^3+b^3>a^2b+ab^2$.

We shall have $$a^3+b^3>a^2b+ab^2,$$
if (dividing each side by $a+b$),
$$a^2-ab+b^2>ab,$$
if $$a^2+b^2>2ab.$$
But this is true (§ 176). $\therefore\ a^3+b^3>a^2b+ab^2.$

(2) Show that $a^2+b^2+c^2>ab+ac+bc$.

Now, $$a^2+b^2>2ab,$$
$$a^2+c^2>2ac, \qquad (§ 176)$$
$$b^2+c^2>2bc.$$
Adding, $$2a^2+2b^2+2c^2>2ab+2ac+2bc.$$
$$\therefore\ a^2+b^2+c^2>ab+ac+bc.$$

Exercise 71.

Show that, the letters being unequal and positive:

1. $a^2+3b^2>2b(a+b)$. 2. $a^3b+ab^3>2a^2b^2$.

3. $(a^2+b^2)(a^4+b^4)>(a^3+b^3)^2$.

4. $a^2b+a^2c+ab^2+b^2c+ac^2+bc^2>6abc$.

5. The sum of any fraction and its reciprocal >2.

6. If $x^2=a^2+b^2$, and $y^2=c^2+d^2$, $xy \not< ac+bd$, or $ad+bc$.

7. $ab+ac+bc<(a+b-c)^2+(a+c-b)^2+(b+c-a)^2$.

8. Which is the greater, $(a^2+b^2)(c^2+d^2)$ or $(ac+bd)^2$?

9. Which is the greater, a^4-b^4 or $4a^3(a-b)$ when $a>b$?

10. Which is the greater, $\sqrt{\frac{a^2}{b}}+\sqrt{\frac{b^2}{a}}$ or $\sqrt{a}+\sqrt{b}$?

11. Which is the greater, $\frac{a+b}{2}$ or $\frac{2ab}{a+b}$?

12. Which is the greater, $\frac{a}{b^2}+\frac{b}{a^2}$ or $\frac{1}{b}+\frac{1}{a}$?

CHAPTER XV.

INVOLUTION AND EVOLUTION.

178. Involution. The operation of raising an expression to any required *power* is called *involution.*

Every case of involution is merely an example of *multiplication,* in which the factors are *equal.*

179. Index Law. If m is a positive integer, by definition

$$a^m = a \times a \times a \cdots\cdots \text{ to } m \text{ factors.} \qquad \S\ 19$$

Consequently, if m and n are both positive integers,

$$\begin{aligned}(a^n)^m &= a^n \times a^n \times a^n \cdots\cdots \text{ to } m \text{ factors.}\\ &= (a \times a \cdots\cdots \text{ to } n \text{ factors})(a \times a \cdots\cdots \text{ to } n \text{ factors})\\ &\qquad \cdots\cdots \text{ taken } m \text{ times}\\ &= a \times a \times a \cdots\cdots \text{ to } mn \text{ factors}\\ &= a^{mn}.\end{aligned}$$

This is the **index law** for involution.

Also,

$$\begin{aligned}(a^m)^n &= a^{mn} = (a^n)^m;\\ (ab)^n &= ab \times ab \cdots\cdots \text{ to } n \text{ factors}\\ &= (a \times a \cdots\cdots \text{ to } n \text{ factors})(b \times b \cdots\cdots \text{ to } n \text{ factors})\\ &= a^n b^n.\end{aligned}$$

180. If the exponent of the required power is a composite number, the exponent may be resolved into prime factors, the power denoted by one of these factors found, and the result raised to a power denoted by another factor of the exponent; and so on. Thus, the fourth power may be obtained by taking the second power of the second power; the sixth by taking the second power of the third power.

181. From the *Law of Signs* in multiplication it is evident that all *even* powers of a number are *positive;* all *odd* powers of a number have the *same sign* as the number itself.

Hence, no *even* power of *any* number can be *negative;* and the even powers of two compound expressions which have the same terms with opposite signs are identical.

Thus, $(b-a)^2=\{-(a-b)\}^2=(a-b)^2.$

182. Binomials. By actual multiplication we obtain,

$$(a+b)^2=a^2+2ab+b^2;$$
$$(a+b)^3=a^3+3a^2b+3ab^2+b^3;$$
$$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4.$$

In these results it will be observed that:

I. The number of terms is greater by one than the exponent of the power to which the binomial is raised.

II. In the first term, the exponent of a is the same as the exponent of the power to which the binomial is raised; and it decreases by one in each succeeding term.

III. b appears in the second term with 1 for an exponent, and its exponent increases by 1 in each succeeding term.

IV. The coefficient of the first term is 1.

V. The coefficient of the second term is the same as the exponent of the power to which the binomial is raised.

VI. The coefficient of each succeeding term is found from the next preceding term by multiplying the coefficient of that term by the exponent of a, and dividing the product by a number greater by one than the exponent of b.

If b is negative, the terms in which the *odd* powers of b occur are negative. Thus,

$$(a-b)^4=a^4-4a^3b+6a^2b^2-4ab^3+b^4.$$

By the above rules any power of a binomial of the form $a+b$, or $a-b$, may be written at once.

183. The same method may be employed when the terms of a binomial have *coefficients* or *exponents*.

(1) $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$

(2) $(5x^2 - 2y^3)^3,$
$$= (5x^2)^3 - 3(5x^2)^2(2y^3) + 3(5x^2)(2y^3)^2 - (2y^3)^3,$$
$$= 125x^6 - 150x^4y^3 + 60x^2y^6 - 8y^9.$$

In like manner, a *polynomial* of three or more terms may be raised to any power by inclosing its terms in parentheses, so as to give the expression the form of a binomial.

(3) $(x^3 - 2x^2 + 3x + 4)^2,$
$$= \{(x^3 - 2x^2) + (3x + 4)\}^2,$$
$$= (x^3 - 2x^2)^2 + 2(x^3 - 2x^2)(3x + 4) + (3x + 4)^2,$$
$$= x^6 - 4x^5 + 4x^4 + 6x^4 - 4x^3 - 16x^2 + 9x^2 + 24x + 16,$$
$$= x^6 - 4x^5 + 10x^4 - 4x^3 - 7x^2 + 24x + 16.$$

Exercise 72.

Perform the indicated operations:

1. $(a^3)^2.$
2. $(x^5)^3.$
3. $(x^2y^3)^2.$
4. $\left(\frac{a^3b^2}{2}\right)^4.$
5. $\left(\frac{3x^2y}{2a^3b^2}\right)^5.$
6. $(x+2)^3.$
7. $(x-2)^4.$
8. $(x+3)^5.$
9. $(1+2x)^5.$
10. $(2m-1)^3.$
11. $(2a^2bc^3)^4.$
12. $(-5ax^3y^2)^3.$
13. $(-7m^3nx^2y^4)^2.$
14. $\left(-\frac{2x^3y}{3abc}\right)^5.$
15. $(3x+1)^4.$
16. $(2x-a)^4.$
17. $(3x+2a)^5.$
18. $(2x-y)^4.$
19. $(x^2y-2xy^2)^6.$
20. $(ab-3)^7.$
21. $(-3a^2b^2c)^5.$
22. $(-3xy^3)^6.$
23. $(-5a^2bx^3)^5.$
24. $\left(-\frac{3ab^2}{4c^3}\right)^4.$
25. $\left(-\frac{x^2y^3z^4}{2}\right)^7.$
26. $(1-a-a^2)^2.$
27. $(2-3x+4x^2)^3.$
28. $(1-2x+x^2)^3.$
29. $(1-x+x^2)^3.$
30. $(1+x+x^2)^4.$

184. Evolution. The operation of finding any required *root* of an expression is called *evolution.*

Every case of evolution is merely an example of *factoring*, in which the required factors are all *equal.* Thus, the square, cube, fourth ..... roots of an expression are found by taking one of the *two*, *three*, *four* ..... *equal factors* of the expression.

The symbol which denotes that a square root is to be extracted is $\sqrt{\ }$; and for other roots the same symbol is used, but with a number-symbol written above to indicate the root; thus, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, signify the *third* root, *fourth* root.

185. Index Law. If m and n are positive integers,

$$(a^m)^n = a^{mn}. \qquad \S\ 179$$

Therefore $$\sqrt[n]{a^{mn}} = a^m.$$

Hence, the root of a simple expression is found by *dividing the exponent of each factor by the index of the root, and taking the product of the resulting factors.*

Thus, the *cube root* of a^6 is a^2; the *fourth root* of $81a^{12}$, that is, 3^4a^{12}, is $3a^3$; and so on.

The above is the **index law** for evolution.

Also, since $$(ab)^n = a^nb^n,$$

therefore, $$\sqrt[n]{a^nb^n} = ab = \sqrt[n]{a^n} \times \sqrt[n]{b^n}.$$

186. It is evident from § 181 that

I. Any *even* root of a *positive* number will have the double sign, $\pm$.

II. There can be no *even* root of a *negative* number.

III. Any *odd* root of a number will have the same sign as the number.

187. The indicated even root of a negative number is called an **impossible,** or **imaginary,** number.

188. If the root of a number expressed in figures is not readily detected, it may be found by resolving the number into its prime factors. Thus, to find the square root of 3,415,104:

$$\begin{array}{r|r} 2^3 & 3415104 \\ \hline 2^3 & 426888 \\ \hline 3^2 & 53361 \\ \hline 7 & 5929 \\ \hline 7 & 847 \\ \hline 11 & 121 \\ \hline & 11 \end{array}$$

$$\therefore \quad 3{,}415{,}104 = 2^6 \times 3^2 \times 7^2 \times 11^2.$$

$$\therefore \quad \sqrt{3415104} = 2^3 \times 3 \times 7 \times 11 = 1848.$$

Exercise 73.

Simplify:

1. $\sqrt{a^4}$, $\sqrt[4]{x^8}$, $\sqrt{4a^6b^2}$, $\sqrt[3]{64}$, $\sqrt[5]{a^5x^{10}y^{15}}$, $\sqrt[4]{16a^{12}b^4c^8}$, $\sqrt[5]{-32a^{15}}$.

2. $\sqrt[3]{-1728c^6d^{12}x^3y^9}$, $\sqrt[3]{3375b^{21}z^{15}}$, $\sqrt[4]{3111696c^{16}z^4}$.

3. $\sqrt{53361b^4c^8y^{12}z^{16}}$, $\sqrt[3]{-\dfrac{216b^3c^{15}}{343z^{24}}}$, $\sqrt[6]{\dfrac{64x^{18}}{729z^{30}}}$.

4. $\sqrt{25a^2b^4c^2} + \sqrt[3]{8a^3b^6c^3} - \sqrt[4]{81a^4b^8c^4} - \sqrt[5]{32a^5b^{10}c^5}$.

5. $\sqrt[3]{27x^3y^6} \times \sqrt[5]{243y^5z^5} \times \sqrt{16x^4z^2}$.

When $a = 1$, $b = 3$, $x = 2$, $y = 6$, find the values of:

6. $4\sqrt{2x} - \sqrt{abxy} + 5\sqrt{a^2b^3xy}$.

7. $2a\sqrt{8ax} + b\sqrt[3]{12by} + 4abx\sqrt{bxy}$.

8. $\sqrt{a^2 + 2ab + b^2} \times \sqrt[3]{a^3 + 3a^2b + 3ab^2 + b^3}$.

9. $\sqrt[3]{b^3 - 3b^2a + 3ba^2 - a^3} \div \sqrt{b^2 + a^2 - 2ab}$.

189. Square Roots of Compound Expressions. Since the square of $a + b$ is $a^2 + 2ab + b^2$, the square root of

$$a^2 + 2ab + b^2 \text{ is } a + b.$$

It is required to devise a method of extracting the square root $a + b$ when $a^2 + 2ab + b^2$ is given.

$$\begin{array}{r|l|l}
 & a^2 + 2ab + b^2 & \underline{a+b} \\
 & \underline{a^2} & \\
2a + b & 2ab + b^2 & \\
 & \underline{2ab + b^2} &
\end{array}$$

The first term, a, of the root is obviously the square root of the first term, a^2, in the expression.

If the a^2 is subtracted from the given expression, the remainder is $2ab + b^2$. Therefore, the second term, b, of the root is obtained by dividing the first term of this remainder by $2a$, that is, by *double the part of the root already found.* Also, since $2ab + b^2 = (2a + b)b$, the divisor is *completed by adding to the trial-divisor the new term of the root.*

The same method will apply to longer expressions, if care be taken to obtain the *trial-divisor* at each stage of the process, by *doubling the part of the root already found*, and to obtain the *complete divisor by annexing the new term of the root to the trial-divisor.*

Find the square root of

$$1 + 10x^2 + 25x^4 + 16x^6 - 24x^5 - 20x^3 - 4x.$$

$$\begin{array}{r|l|l}
 & 16x^6 - 24x^5 + 25x^4 - 20x^3 + 10x^2 - 4x + 1 & \underline{4x^3 - 3x^2 + 2x - 1} \\
 & \underline{16x^6} & \\
8x^3 - 3x^2 & -24x^5 + 25x^4 & \\
 & \underline{-24x^5 + 9x^4} & \\
8x^3 - 6x^2 + 2x & 16x^4 - 20x^3 + 10x^2 & \\
 & \underline{16x^4 - 12x^3 + 4x^2} & \\
8x^3 - 6x^2 + 4x - 1 & -8x^3 + 6x^2 - 4x + 1 & \\
 & \underline{-8x^3 + 6x^2 - 4x + 1} &
\end{array}$$

The expression is arranged according to descending powers of x.

It will be noticed that each successive trial-divisor may be obtained by taking the preceding complete divisor with its *last term doubled.*

Exercise 74.

Extract the square root of:

1. $a^4 + 4a^3 + 2a^2 - 4a + 1.$

2. $x^4 - 2x^3y + 3x^2y^2 - 2xy^3 + y^4.$

3. $4a^6 - 12a^5x + 5a^4x^2 + 6a^3x^3 + a^2x^4.$

4. $9x^6 - 12x^3y^3 + 16x^2y^4 - 24x^4y^2 + 4y^6 + 16xy^5.$

5. $4a^8 + 16c^8 + 16a^6c^2 - 32a^2c^6.$

6. $4x^4 + 9 - 30x - 20x^3 + 37x^2.$

7. $16x^4 - 16abx^2 + 16b^2x^2 + 4a^2b^2 - 8ab^3 + 4b^4.$

8. $x^6 + 25x^2 + 10x^4 - 4x^5 - 20x^3 + 16 - 24x.$

9. $x^6 + 8x^4y^2 - 4x^5y - 4xy^5 + 8x^2y^4 - 10x^3y^3 + y^6.$

10. $4 - 12a - 11a^4 + 5a^2 - 4a^5 + 4a^6 + 14a^3.$

11. $9a^2 - 6ab + 30ac + 6ad + b^2 - 10bc - 2bd$
$+ 25c^2 + 10cd + d^2.$

12. $25x^6 - 31x^4y^2 + 34x^3y^3 - 30x^5y + y^6 - 8xy^5 + 10x^2y^4.$

13. $m^8 - 4m^7 + 10m^6 - 20m^5 - 44m^3 + 35m^4$
$+ 46m^2 - 40m + 25.$

14. $x^4 - x^3y - \frac{7}{4}x^2y^2 + xy^3 + y^4.$

15. $x^4 - 4x^3y + 6x^2y^2 - 6xy^3 + 5y^4 - \dfrac{2y^5}{x} + \dfrac{y^6}{x^2}.$

16. $\dfrac{a^4}{9} - \dfrac{a^3x}{2} + \frac{43}{48}a^2x^2 - \frac{3}{4}ax^3 + \dfrac{x^4}{4}.$

17. $1 + \dfrac{4}{x} + \dfrac{10}{x^2} + \dfrac{20}{x^3} + \dfrac{25}{x^4} + \dfrac{24}{x^5} + \dfrac{16}{x^6}.$

18. $\dfrac{a^2}{b^2} - \dfrac{2a}{b} + 3 - \dfrac{2b}{a} + \dfrac{b^2}{a^2}.$

19. $x^4 + x^3 - \dfrac{5x^2}{12} - \dfrac{x}{3} + \dfrac{1}{9}.$

190. Arithmetical Square Roots. In the general method of extracting the square root of a number expressed by figures, the first step is to divide the figures into groups.

Since $1 = 1^2$, $100 = 10^2$, $10{,}000 = 100^2$, and so on, it is evident that the square root of any integral square number between 1 and 100 lies between 1 and 10; the square root of any integral square number between 100 and 10,000 lies between 10 and 100. In other words, the square root of any integral square number expressed by *one* or *two* figures is a number of *one* figure; the square root of any integral square number expressed by *three* or *four* figures is a number of *two* figures; and so on.

If, therefore, an integral square number be divided into groups of two figures each, from the right to the left, the number of figures in the root will be equal to the number of groups of figures. The last group to the left may consist of only one figure.

Find the square root of 3249.

$$\begin{array}{r|l} & 32\,49(57 \\ & \underline{25} \\ 107) & 7\,49 \\ & \underline{7\,49} \end{array}$$

In this case, a in the typical form $a^2 + 2ab + b^2$ represents 5 *tens*, that is, 50, and b represents 7. The 25 subtracted is really 2500, that is, a^2, and the complete divisor $2a + b$ is $2 \times 50 + 7 = 107$.

The same method will apply to numbers of more than two groups by considering that a in the typical form represents at each step *the part of the root already found*, and that a represents *tens* with reference to the next figure of the root.

191. If the square root of a number have decimal places, the number itself will have *twice* as many.

Thus, if 0.11 be the square root of some number, the number will be $(0.11)^2 = 0.11 \times 0.11 = 0.0121$. Hence, if a given number contains a decimal, we divide it into groups of two figures each, by beginning at the decimal point and proceeding toward the left for the integral number, and toward the right for the decimal. We must be careful to have the last group on the right of the decimal point contain *two* figures, annexing a cipher when necessary.

192. If a number contains an *odd* number of decimal places, or gives a *remainder* when as many figures in the root have been obtained as the given number has groups, then its exact square root cannot be found. We may, however, approximate to the exact root as near as we please by annexing ciphers and continuing the operation.

Find the square roots of 3 and 357.357.

$$\begin{array}{r|l}
 & 3.(1.732\ldots\ldots \\
 & 1 \\
\hline
27) & 2\,00 \\
 & 1\,89 \\
\hline
343) & 11\,00 \\
 & 10\,29 \\
\hline
3462) & 71\,00 \\
 & 69\,24 \\
\hline
\end{array}
\qquad\qquad
\begin{array}{r|l}
 & 3\,57.35\,70(18.903\ldots\ldots \\
 & 1 \\
\hline
28) & 2\,57 \\
 & 2\,24 \\
\hline
369) & 33\,35 \\
 & 33\,21 \\
\hline
37803) & 14\,70\,00 \\
 & 11\,34\,09 \\
\hline
\end{array}$$

193. The square root of a common fraction is found by extracting the square root of the numerator and the square root of the denominator. But, when the denominator is not a perfect square, it is best to reduce the fraction to a decimal and then extract the root.

Exercise 75.

Extract the square root of:

1. 120,409; 4816.36; 1867.1041; 1435.6521; 64.128064.
2. 16,803.9369; 4.54499761; 0.24373969; 0.5687573056.
3. 0.9; 6.21; 0.43; 0.00852; 17; 129; 347.259.
4. 14,295.387; 2.5; 2000; 0.3; 0.03; 111.
5. 0.00111; 0.004; 0.005; 2; 5; 3.25; 8.6.
6. $\frac{1}{4}$; $\frac{16}{49}$; $\frac{100}{144}$; $\frac{169}{225}$; $\frac{289}{324}$; $\frac{400}{625}$.
7. $\frac{1}{2}$; $\frac{2}{3}$; $\frac{3}{4}$; $\frac{1}{32}$; $\frac{7}{128}$; $\frac{6}{125}$; $\frac{6}{7}$; $\frac{1}{12}$.

194. Cube Roots of Compound Expressions. Since the cube of $a+b$ is $a^3+3a^2b+3ab^2+b^3$, the cube root of

$$a^3+3a^2b+3ab^2+b^3 \text{ is } a+b.$$

It is required to devise a method for extracting the cube root $a+b$ when $a^3+3a^2b+3ab^2+b^3$ is given.

$$\begin{array}{l|l}
 & a^3+3a^2b+3ab^2+b^3 \ \underline{|\,a+b} \\
3a^2 & a^3 \\
\quad +3ab+b^2 & \quad 3a^2b+3ab^2+b^3 \\
\hline
3a^2+3ab+b^2 & \quad 3a^2b+3ab^2+b^3 \\
\end{array}$$

The first term of the root is a, the cube root of a^3.

If a^3 is subtracted, the remainder is $3a^2b+3ab^2+b^3$; therefore, the second term b of the root is obtained by dividing the first term of this remainder by *three times the square of a.*

Also, since $3a^2b+3ab^2+b^3=(3a^2+3ab+b^2)b$, the *complete divisor* is obtained by adding $3ab+b^2$ to the *trial-divisor* $3a^2$.

The same method may be applied to longer expressions by considering a in the typical form to represent at each stage of the process *the part of the root already found.*

Find the cube root of $x^6-3x^5+5x^3-3x-1$.

$$\begin{array}{rl|l}
 & & \underline{|\,x^2-x-1} \\
 & & x^6-3x^5+5x^3-3x-1 \\
 & 3x^4 & x^6 \\
(3x^2-x)(-x)= & \quad -3x^3+x^2 & -3x^5+5x^3 \\
 & 3x^4-3x^3+x^2 & -3x^5+3x^4-x^3 \\
\hline
 & & -3x^4+6x^3-3x-1 \\
3(x^2-x)^2= & 3x^4-6x^3+3x^2 & \\
(3x^2-3x-1)(-1)= & \quad -3x^2+3x+1 & \\
 & 3x^4-6x^3+3x+1 & -3x^4+6x^3-3x-1 \\
\end{array}$$

The first trial-divisor is $3x^4$, and the first complete divisor is $3x^4-3x^3+x^2$. The second trial-divisor is $3(x^2-x)^2$, or $3x^4-6x^3+3x^2$. The second term of the root is found by dividing $-3x^5$, the first term of the remainder, by $3x^4$, the first term of the root. The second complete divisor is $3x^4-6x^3+3x+1$.

Exercise 76.

Find the cube root of:

1. $x^3 + 6x^2y + 12xy^2 + 8y^3$.
2. $a^3 - 9a^2 + 27a - 27$.
3. $x^3 + 12x^2 + 48x + 64$.
4. $x^6 - 3ax^5 + 5a^3x^3 - 3a^5x - a^6$.
5. $x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 3x + 1$.
6. $1 - 9x + 39x^2 - 99x^3 + 156x^4 - 144x^5 + 64x^6$.
7. $a^6 - 6a^5 + 9a^4 + 4a^3 - 9a^2 - 6a - 1$.
8. $64x^6 + 192x^5 + 144x^4 - 32x^3 - 36x^2 + 12x - 1$.
9. $1 - 3x + 6x^2 - 10x^3 + 12x^4 - 12x^5 + 10x^6 - 6x^7 + 3x^8 - x^9$.
10. $a^6 + 9a^5b - 135a^3b^3 + 729ab^5 - 729b^6$.
11. $c^6 - 12bc^5 + 60b^2c^4 - 160b^3c^3 + 240b^4c^2 - 192b^5c + 64b^6$.
12. $8a^6 + 48a^5b + 60a^4b^2 - 80a^3b^3 - 90a^2b^4 + 108ab^5 - 27b^6$.

195. Arithmetical Cube Roots. In extracting the cube root of a number expressed by figures, the first step is to divide it into groups.

Since $1 = 1^3$, $1000 = 10^3$, $1,000,000 = 100^3$, and so on, it follows that the cube root of any integral cube number between 1 and 1000, that is, of any integral cube number which has *one*, *two*, or *three* figures, is a number of *one* figure; and that the cube root of any integral cube number between 1000 and 1,000,000, that is, of any integral cube number which has *four*, *five*, or *six* figures, is a number of *two* figures; and so on.

If, therefore, an integral cube number be divided into groups of three figures each, from right to left, the number of figures in the root will be equal to the number of groups. The last group to the left may consist of one, two, or three figures.

196. If the cube root of a number have decimal places, the number itself will have *three times* as many.

Hence, if a given number contains a decimal, we divide the figures of the number into groups of three figures each, beginning at the decimal-point and proceeding toward the left for the integral number, and toward the right for the decimal. We must annex ciphers if necessary, so that the last group on the right may contain *three* figures.

If the given number is not a perfect cube, zeros may be annexed, and an approximate value of the root found.

197. In the typical form, the *first complete divisor* is

$$3a^2 + 3ab + b^2;$$

the *second trial-divisor* is $3(a+b)^2$, or $3a^2 + 6ab + 3b^2$, which may be obtained by adding to the preceding complete divisor *its second term and twice its third term.*

Extract the cube root of 5 to five places of decimals.

		5.000(1.70997
		1
$3 \times 10^2 =$	300	4000
$3(10 \times 7) =$	210	
$7^2 =$	49	
	559	3913
	259	87000000
$3 \times 1700^2 =$	8670000	
$3(1700 \times 9) =$	45900	
$9^2 =$	81	
	8715981	78443829
	45981	85561710
$3 \times 1709^2 =$	8762043	78858387
		67033230
		61334301

After the first two figures of the root are found, the next trial-divisor is obtained by bringing down the sum of the 210 and 49 obtained in completing the preceding divisor; then adding the three lines connected by the brace, and annexing two ciphers to the result.

The last two figures of the root are found by division. The rule in such cases is, that two less than the number of figures already obtained may be found without error by division, the divisor being three times the square of the part of the root already found.

Exercise 77.

Find the cube root of:

1. 274,625.
2. 110,592.
3. 262,144.
4. 884,736.
5. 109,215,352.
6. 1,481,544.
7. 1601.613.
8. 1,259,712.
9. 2.803221.
10. 7,077,888.
11. 12.812904.
12. 56.623104.
13. 33,076.161.
14. 102,503.232.
15. 820.025856.
16. 8653.002877.
17. 1.371330631.
18. 20,910.518875.
19. 91.398648466125.
20. 5.340104393239.
21. Find to four figures the cube roots of 2.5; 0.2; 0.01; 4; 0.4.

198. Since the fourth power is the square of the square, and the sixth power the square of the cube; the *fourth root* is the *square root* of the *square root*, and the *sixth root* is the *cube root* of the *square root*. In like manner, the eighth, ninth, twelfth..... roots may be found.

Exercise 78.

Find the fourth root of:

1. $81a^4 - 540a^3b + 1350a^2b^2 - 1500ab^3 + 625b^4$.
2. $1 - 4x + 10x^2 - 16x^3 + 19x^4 - 16x^5 + 10x^6 - 4x^7 + x^8$.

Find the sixth root of:

3. $64 - 192x + 240x^2 - 160x^3 + 60x^4 - 12x^5 + x^6$.
4. $729x^6 - 1458x^5 + 1215x^4 - 540x^3 + 135x^2 - 18x + 1$.

Find the eighth root of:

5. $1 - 8y + 28y^2 - 56y^3 + 70y^4 - 56y^5 + 28y^6 -- 8y^7 + y^8$.

CHAPTER XVI.

THEORY OF EXPONENTS.

199. If n is a positive integer, we have defined a^n to mean the product obtained by taking a as a factor n times. Thus a^3 stands for $a \times a \times a$; b^4 stands for $b \times b \times b \times b$.

200. From this definition we have obtained the following laws for positive and integral exponents:

$$\text{I.} \quad a^m \times a^n = a^{m+n}.$$

$$\text{II.} \quad (a^m)^n = a^{mn}.$$

$$\text{III.} \quad \frac{a^m}{a^n} = a^{m-n}, \text{ if } m > n.$$

$$\text{IV.} \quad \sqrt[n]{a^{mn}} = a^m.$$

$$\text{V.} \quad (ab)^n = a^n b^n.$$

201. Since by the definition of a^n the exponent n denotes simply *repetitions* of a as a factor, such expressions as $a^{\frac{2}{3}}$ and a^{-3} have no meaning whatever. It is found convenient, however, to extend the meaning of a^n so as to include fractional and negative values of n.

202. If we do not define the meaning of a^n when n is a fraction or negative, but require that the meaning of a^n must in all cases be such that the fundamental index law shall always hold true, namely,

$$a^m \times a^n = a^{m+n},$$

we shall find that this condition alone will be sufficient to define the meaning of a^n for all cases.

203. To find the Meaning of a Fractional Exponent.

Assuming the index law to hold true for fractional exponents, we have

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^{\frac{2}{2}} = a,$$
$$a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^{\frac{3}{3}} = a,$$
$$a^{\frac{3}{4}} \times a^{\frac{3}{4}} \times a^{\frac{3}{4}} \times a^{\frac{3}{4}} = a^{\frac{3}{4}+\frac{3}{4}+\frac{3}{4}+\frac{3}{4}} = a^{\frac{12}{4}} = a^3,$$
$$a^{\frac{1}{n}} \times a^{\frac{1}{n}} \cdots\cdots \text{ to } n \text{ factors} = a^{\frac{1}{n}+\frac{1}{n} \cdots\cdots \text{ to } n \text{ terms}} = a^{\frac{n}{n}} = a,$$
$$a^{\frac{m}{n}} \times a^{\frac{m}{n}} \cdots\cdots \text{ to } n \text{ factors} = a^{\frac{m}{n}+\frac{m}{n} \cdots\cdots \text{ to } n \text{ terms}} = a^{\frac{mn}{n}} = a^m.$$

That is, $a^{\frac{1}{2}}$ is one of the two equal factors of a,

$a^{\frac{1}{3}}$ is one of the three equal factors of a,

$a^{\frac{3}{4}}$ is one of the four equal factors of a^3,

$a^{\frac{1}{n}}$ is one of the n equal factors of a,

$a^{\frac{m}{n}}$ is one of the n equal factors of a^m.

Hence, $a^{\frac{1}{2}} = \sqrt{a}$; $a^{\frac{1}{3}} = \sqrt[3]{a}$;

$a^{\frac{3}{4}} = \sqrt[4]{a^3}$; $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.

Also, $a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \cdots\cdots$ to m factors,

$$= a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n} \cdots\cdots \text{ to } m \text{ terms}} = a^{\frac{m}{n}}.$$
$$\therefore\ a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m.$$

The meaning, therefore, of $a^{\frac{m}{n}}$, where m and n are positive integers, is, the nth root of the mth power of a, or the mth power of the nth root of a.

Hence the numerator of a fractional exponent indicates a power, and the denominator a root; and the result is the same when we first extract the root and raise this root to the required power, as when we first find the power and extract the required root of this power.

204. To find the Meaning of a^0.

By the index law,

$$a^0 \times a^m = a^{0+m} = a^m.$$
$$\therefore\ a^0 = a^m \div a^m.$$
$$\therefore\ a^0 = 1, \text{ whatever the value of } a \text{ is.}$$

205. To find the Meaning of a Negative Exponent.

If n stands for a positive integer, or a positive fraction, we have by the index law,

$$a^n \times a^{-n} = a^{n-n} = a^0.$$

But
$$a^0 = 1.$$
$$\therefore\ a^n \times a^{-n} = 1.$$

That is, a^n and a^{-n} are *reciprocals* of each other (§ 149), so that $a^{-n} = \dfrac{1}{a^n}$, and $a^n = \dfrac{1}{a^{-n}}$.

206. Hence, we can change any *factor* from the numerator of a fraction to the denominator, or from the denominator to the numerator, *provided we change the sign of its exponent.*

Thus $\dfrac{ab^2}{c^3d^3}$ may be written $ab^2c^{-3}d^{-3}$, or $\dfrac{1}{a^{-1}b^{-2}c^3d^3}$.

207. We have now assigned definite meanings to fractional and negative exponents, meanings obtained by subjecting them to the fundamental index law of positive integral exponents; and we will now show that Law II., namely, $(a^m)^n = a^{mn}$, which has been established for positive integral exponents, holds true for fractional and negative exponents.

(1) If n is a positive integer, whatever the value of m, we have

$$(a^m)^n = a^m \times a^m \times a^m \cdots\cdots \text{ to } n \text{ factors,}$$
$$= a^{m+m+m \cdots\cdots \text{ to } n \text{ terms}},$$
$$= a^{mn}.$$

(2) If n is a positive fraction $\frac{p}{q}$, where p and q are positive integers, we have

$$(a^m)^n = (a^m)^{\frac{p}{q}} = \sqrt[q]{(a^m)^p} \qquad \text{§ 203}$$
$$= \sqrt[q]{a^{mp}} \qquad (1)$$
$$= a^{\frac{mp}{q}} \qquad \text{§ 203}$$
$$= a^{m \times \frac{p}{q}}$$
$$= a^{mn}.$$

(3) If n is a negative integer, and equal to $-p$, we have

$$(a^m)^n = (a^m)^{-p} = \frac{1}{(a^m)^p} \qquad \text{§ 205}$$
$$= \frac{1}{a^{mp}} \qquad (1)$$
$$= a^{-mp} \qquad \text{§ 205}$$
$$= a^{m(-p)}$$
$$= a^{mn}.$$

(4) If n is negative and equal to the fraction $-\frac{p}{q}$, where p and q are positive integers, we have

$$(a^m)^n = (a^m)^{-\frac{p}{q}} = \frac{1}{(a^m)^{\frac{p}{q}}} \qquad \text{§ 205}$$
$$= \frac{1}{\sqrt[q]{a^{mp}}} \qquad \text{§ 203}$$
$$= \frac{1}{a^{\frac{mp}{q}}} \qquad \text{§ 203}$$
$$= \frac{1}{a^{m \times \frac{p}{q}}}$$
$$= \frac{1}{a^{m(-n)}}$$
$$= \frac{1}{a^{-mn}} \qquad (1)$$
$$= a^{mn}. \qquad \text{§ 205}$$

Hence, $(a^m)^n = a^{mn}$, for *all values* of m and n.

208. In like manner it may be shown that all the index laws of positive integral exponents apply also to fractional, and negative, exponents.

Exercise 79.

Express with fractional exponents:

1. $\sqrt{x^3}$; $\sqrt[3]{x^2}$; $(\sqrt{x})^5$; $\sqrt[3]{a^4}$; $\sqrt[7]{a^6}$; $(\sqrt[3]{a})^7$; $\sqrt[6]{a^3b^2}$.

2. $\sqrt[3]{xy^2z^3}$; $\sqrt[5]{x^3y^2z^4}$; $\sqrt[7]{a^5b^6c^7}$; $5\sqrt{a^2bc^3x^4}$.

Express with radical signs:

3. $a^{\frac{2}{3}}$; $a^{\frac{1}{2}}b^{\frac{1}{6}}$; $4x^{\frac{1}{6}}y^{-\frac{5}{6}}$; $3x^{\frac{1}{3}}y^{-\frac{2}{3}}$.

Express with positive exponents:

4. a^{-2}; $3x^{-1}y^{-3}$; $6x^{-3}y$; x^4y^{-5}; $\dfrac{2a^{-1}x}{3^{-1}b^2y^{-3}}$.

Write in the form of integral expressions:

5. $\dfrac{3xy}{z^2}$; $\dfrac{z}{x^3y^4}$; $\dfrac{a}{bc}$; $\dfrac{c^2}{a^3b^{-2}}$; $\dfrac{x^{-\frac{1}{3}}}{y^{-\frac{2}{3}}}$; $\dfrac{x^{\;2}}{y^{\frac{1}{3}}}$.

Simplify:

6. $a^{\frac{1}{2}} \times a^{\frac{1}{3}}$; $b^{\frac{1}{3}} \times b^{\frac{1}{6}}$; $c^{\frac{2}{3}} \times c^{\frac{1}{12}}$; $d^{\frac{3}{5}} \times d^{\frac{1}{15}}$.

7. $m^{\frac{1}{2}} \times m^{-\frac{1}{6}}$; $n^{\frac{3}{4}} \times n^{-\frac{1}{12}}$; $a^0 \times a^{\frac{1}{2}}$; $a^0 \times a^{-\frac{1}{2}}$.

8. $a^{\frac{1}{2}} \times \sqrt{a}$; $c^{-\frac{1}{2}} \times \sqrt{c}$; $y^{\frac{1}{4}} \times \sqrt[4]{y}$; $x^{\frac{5}{8}} \times \sqrt{x^{-1}}$.

9. $ab^{\frac{1}{2}}c \times a^{-\frac{1}{2}}bc^{\frac{1}{3}}$; $a^{\frac{2}{3}}b^{\frac{1}{2}}c^{-\frac{1}{4}} \times a^{\frac{1}{3}}b^{-\frac{1}{2}}c^{\frac{1}{2}}d$.

10. $x^{\frac{1}{3}}y^{\frac{2}{3}}z^{\frac{1}{6}} \times x^{-\frac{2}{3}}y^{-\frac{1}{2}}z^{-\frac{1}{2}}$; $x^{\frac{5}{8}}y^{\frac{1}{4}}z^{\frac{1}{2}} \times x^{-\frac{1}{8}}y^{-\frac{1}{2}}z^{-\frac{1}{2}}$.

11. $a^{\frac{1}{2}} \times a^{-\frac{1}{3}} \times a^{-\frac{1}{4}} \times a^{-\frac{1}{3}}$; $\left(\dfrac{ay}{x}\right)^{\frac{1}{2}} \times \left(\dfrac{bx}{y^2}\right)^{\frac{1}{3}} \times \left(\dfrac{y^2}{a^2b^2}\right)^{\frac{1}{4}}$.

12. $a^{\frac{1}{2}} \div a^{\frac{1}{3}}$; $c^{\frac{5}{8}} \div c^{\frac{1}{2}}$; $n^{\frac{7}{12}} \div n^{\frac{3}{4}}$; $a^{\frac{5}{6}} \div \sqrt[3]{a^2}$.

13. $(a^6)^{\frac{1}{2}} \div (a^6)^{\frac{2}{3}}$; $(c^{-\frac{1}{2}})^{\frac{2}{5}}$; $(m^{-\frac{1}{2}})^4$; $(n^{\frac{1}{3}})^{-3}$; $(x^{\frac{3}{4}})^{\frac{4}{3}}$.

14. $(p^{-\frac{3}{4}})^{-\frac{2}{3}}$; $(q^{\frac{2}{5}})^{-\frac{1}{2}}$; $(x^{-\frac{3}{8}}y^{\frac{9}{4}})^{-\frac{4}{9}}$; $(a^{\frac{2}{3}} \times a^{\frac{4}{7}})^{-\frac{14}{13}}$.

15. $(4a^{-\frac{2}{3}})^{-\frac{3}{2}}$; $(27b^{-3})^{-\frac{2}{3}}$; $(64c^{16})^{-\frac{5}{6}}$; $(32c^{-10})^{\frac{2}{5}}$.

16. $\left(\dfrac{16a^{-4}}{81b^3}\right)^{-\frac{3}{4}}$; $\left(\dfrac{9a^4}{16b^{-3}}\right)^{-\frac{3}{2}}$; $(3^{\frac{3}{8}}a^{-3})^{-\frac{2}{3}}$; $(\frac{256}{625})^{-\frac{3}{4}}$.

209. Compound Expressions having fractional or negative exponents are multiplied and divided the same as compound expressions having positive integral exponents.

(1) Multiply $y^{\frac{3}{4}} + y^{\frac{1}{2}} + y^{\frac{1}{4}} + 1$ by $y^{\frac{1}{4}} - 1$.

$$\begin{array}{l}
y^{\frac{3}{4}} + y^{\frac{1}{2}} + y^{\frac{1}{4}} + 1 \\
\underline{y^{\frac{1}{4}} - 1} \\
y + y^{\frac{3}{4}} + y^{\frac{1}{2}} + y^{\frac{1}{4}} \\
\underline{\quad - y^{\frac{3}{4}} - y^{\frac{1}{2}} - y^{\frac{1}{4}} - 1} \\
y \qquad\qquad\qquad\quad - 1
\end{array}
\qquad\qquad y - 1. \ \textit{Ans.}$$

(2) Divide $x^{\frac{2}{3}} + x^{\frac{1}{3}} - 12$ by $x^{\frac{1}{3}} - 3$.

$$\begin{array}{l|l}
x^{\frac{2}{3}} + x^{\frac{1}{3}} - 12 & x^{\frac{1}{3}} - 3 \\ \cline{2-2}
\underline{x^{\frac{2}{3}} - 3x^{\frac{1}{3}}} & x^{\frac{1}{3}} + 4 \\
\quad 4x^{\frac{1}{3}} - 12 & \\
\quad \underline{4x^{\frac{1}{3}} - 12} &
\end{array}
\qquad\qquad x^{\frac{1}{3}} + 4. \ \textit{Ans.}$$

Exercise 80.

Multiply:

1. $x^{2p} + x^p y^p + y^{2p}$ by $x^{2p} - x^p y^p + y^{2p}$.
2. $x^{mn-n} - y^n$ by $x^n + y^{mn-n}$.
3. $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} + 1$ by $x^{\frac{1}{3}} - 1$.
4. $8a^{\frac{3}{7}} + 4a^{\frac{2}{7}}b^{\frac{1}{7}} + 5a^{\frac{1}{7}}b^{\frac{2}{7}} + 9b^{\frac{3}{7}}$ by $2a^{\frac{4}{7}} - b^{\frac{4}{7}}$.

5. $1+ab^{-1}+a^2b^{-2}$ by $1-ab^{-1}+a^2b^{-2}$.

6. $a^2b^{-2}+2+a^{-2}b^2$ by $a^2b^{-2}-2-a^{-2}b^2$.

7. $4x^{-3}+3x^{-2}+2x^{-1}+1$ by $x^{-2}-x^{-1}+1$.

Divide:

8. $x^{4n}-y^{4n}$ by x^n-y^n.

9. $x+y+z-3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}$ by $x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}}$.

10. $x+y$ by $x^{\frac{4}{5}}-x^{\frac{3}{5}}y^{\frac{1}{5}}+x^{\frac{2}{5}}y^{\frac{2}{5}}-x^{\frac{1}{5}}y^{\frac{3}{5}}+y^{\frac{4}{5}}$.

11. $x^2y^{-2}+2+x^{-2}y^2$ by $xy^{-1}+x^{-1}y$.

12. $a^{-4}+a^{-2}b^{-2}+b^{-4}$ by $a^{-2}-a^{-1}b^{-1}+b^{-2}$.

Find the squares of:

13. $4ab^{-1}$; $a^{\frac{1}{2}}-b^{\frac{1}{2}}$; $a+a^{-1}$; $2a^{\frac{1}{2}}b^{\frac{1}{3}}-a^{-\frac{1}{2}}b^{\frac{2}{3}}$.

If $a=4$, $b=2$, $c=1$, find the values of:

14. $a^{\frac{1}{2}}b$; $5ab^{-1}$; $2(ab)^{\frac{1}{3}}$; $a^{-\frac{1}{2}}b^{-1}c^{\frac{2}{3}}$; $12a^{-2}b^{-3}$.

15. Expand $(a^{\frac{1}{2}}-b^{\frac{1}{3}})^3$; $(2x^{-1}+x)^4$; $(ab^{-1}-by^{-1})^6$.

Extract the square root of:

16. $9x^{-4}-18x^{-3}y^{\frac{1}{2}}+15x^{-2}y-6x^{-1}y^{\frac{3}{2}}+y^2$.

Extract the cube root of:

17. $8x^3+12x^2-30x-35+45x^{-1}+27x^{-2}-27x^{-3}$.

Resolve into prime factors with fractional exponents:

18. $\sqrt[3]{12}$, $\sqrt[4]{72}$, $\sqrt[6]{96}$, $\sqrt[8]{64}$; and find their product.

Simplify:

19. $\{(x^{5ab})^3 \times (x^{5b})^{-2}\}^{\frac{1}{3a-2}}$. 20. $(x^{18a} \times x^{-12})^{\frac{1}{3a-2}}$.

21. $3(a^{\frac{1}{2}}+b^{\frac{1}{2}})^2-4(a^{\frac{1}{2}}+b^{\frac{1}{2}})(a^{\frac{1}{2}}-b^{\frac{1}{2}})+(a^{\frac{1}{2}}-2b^{\frac{1}{2}})^2$.

22. $\{(a^m)^{m-\frac{1}{m}}\}^{\frac{1}{m+1}}$. 24. $[\{(a^{-m})^{-n}\}^p]^q \div [\{(a^m)^n\}^{-p}]^{-q}$.

23. $\left(\dfrac{x^{p+q}}{x^q}\right)^p \div \left(\dfrac{x^q}{x^{q-p}}\right)^{p-q}$. 25. $\dfrac{x^{2p(q-1)}-y^{2q(p-1)}}{x^{p(q-1)}+y^{q(p-1)}}$.

CHAPTER XVII.

RADICAL EXPRESSIONS.

210. A **radical expression** is an expression affected with the radical sign; as, $\sqrt{a}$, $\sqrt[6]{9}$, $\sqrt[3]{a^2}$, $\sqrt[4]{a+b}$, $\sqrt[5]{32}$.

211. An indicated root that cannot be exactly obtained is called a **surd.** An indicated root that can be exactly obtained is said to have the *form* of a surd.

The required root shows the **order** of a surd; and surds are named quadratic, cubic, biquadratic, according as the second, third, or fourth roots are required.

The product of a rational factor and a surd factor is called a **mixed surd;** as $3\sqrt{2}$, $b\sqrt{a}$. The rational factor of a mixed surd is called the **coefficient** of the radical.

When there is no rational factor outside of the radical sign, that is, when the coefficient is 1, the surd is said to be **entire;** as, $\sqrt{2}$, $\sqrt{a}$.

212. A surd is in its *simplest form* when the expression under the radical sign is *integral and as small as possible.*

Surds which have the same surd factor, when reduced to the simplest form, are said to be **similar.**

NOTE. In operations with surds, arithmetical numbers contained in the surds should be expressed in their prime factors.

REDUCTION OF RADICALS.

213. To reduce a radical is to change its *form* without changing its *value.*

CASE I.

214. When the Radical is a Perfect Power and has for an Exponent a Factor of the Index of the Root.

(1) $\sqrt[4]{a^2} = a^{\frac{2}{4}} = a^{\frac{1}{2}} = \sqrt{a}$;

(2) $\sqrt[4]{36\,a^2b^2} = \sqrt[4]{(6\,ab)^2} = (6\,ab)^{\frac{2}{4}} = (6\,ab)^{\frac{1}{2}} = \sqrt{6\,ab}$;

(3) $\sqrt[6]{25\,a^4b^2c^8} = \sqrt[6]{(5\,a^2bc^4)^2} = (5\,a^2bc^4)^{\frac{2}{6}} = (5\,a^2bc^4)^{\frac{1}{3}}$
$= \sqrt[3]{5\,a^2bc^4}$.

We have, therefore, the following rule:

Divide the exponent of the power by the index of the root.

Exercise 81.

Simplify:

1. $\sqrt[4]{36}$.
2. $\sqrt[8]{81}$.
3. $\sqrt[6]{125}$.
4. $\sqrt[4]{100}$.
5. $\sqrt[6]{343}$.
6. $\sqrt[6]{4\,a^2b^2}$.
7. $\sqrt[4]{9\,a^2b^2}$.
8. $\sqrt[8]{16\,a^4b^4}$.
9. $\sqrt[6]{343\,a^3b^6}$.
10. $\sqrt[8]{81\,a^4b^4}$.
11. $\sqrt[6]{\dfrac{36\,c^2}{49\,a^2}}$.
12. $\sqrt[4]{\dfrac{(x-5)^2}{(x+3)^2}}$.
13. $\sqrt[6]{\dfrac{8\,a^3b^6}{27\,x^3y^3}}$.

CASE II.

215. When the Radical is the Product of Two Factors, One of which is a Perfect Power of the Same Degree as the Radical.

Since $\sqrt[n]{a^nb} = \sqrt[n]{a^n} \times \sqrt[n]{b} = a\sqrt[n]{b}$ (§ 185), we have

(1) $\sqrt{a^2b} = \sqrt{a^2} \times \sqrt{b} = a\sqrt{b}$;

(2) $\sqrt[3]{108} = \sqrt[3]{27 \times 4} = \sqrt[3]{27} \times \sqrt[3]{4} = 3\sqrt[3]{4}$;

(3) $4\sqrt{72a^2b^3} = 4\sqrt{36a^2b^2 \times 2b} = 4\sqrt{36a^2b^2} \times \sqrt{2b}$
$= 4 \times 6ab\sqrt{2b} = 24ab\sqrt{2b}$;

(4) $2\sqrt[3]{54a^4b} = 2\sqrt[3]{27a^3 \times 2ab} = 2\sqrt[3]{27a^3} \times \sqrt[3]{2ab}$
$= 2 \times 3a\sqrt[3]{2ab} = 6a\sqrt[3]{2ab}$.

We have, therefore, the following rule:

Resolve the radical into two factors, one of which is the greatest perfect power of the same degree as the radical.

Remove this factor from under the radical sign, extract the required root, and multiply the coefficient of the radical by the root obtained.

Exercise 82.

Simplify:

1. $\sqrt{125}$.
2. $\sqrt{243}$.
3. $\sqrt[3]{162}$.
4. $\sqrt[3]{256}$.
5. $\sqrt[3]{375}$.
6. $\sqrt[3]{320}$.
7. $\sqrt[5]{486}$.
8. $\sqrt[4]{729}$.
9. $\sqrt[4]{208}$.
10. $\sqrt{605}$.
11. $2\sqrt[4]{144}$.
12. $3\sqrt[3]{2662}$.
13. $7\sqrt[4]{176}$.
14. $7\sqrt{m^2n}$.
15. $5\sqrt[4]{b^8a^5}$.
16. $6\sqrt[5]{a^{18}c^8}$.
17. $3\sqrt[6]{a^{12}b^{19}}$.
18. $7\sqrt[3]{64a^3b}$.
19. $6\sqrt[3]{108m^2n^3}$.
20. $4\sqrt[4]{x^{11}y^{12}}$.
21. $2\sqrt[3]{-1029}$.
22. $\sqrt[3]{-1458}$.
23. $3\sqrt[4]{1875}$.
24. $4\sqrt[3]{686}$.
25. $\sqrt[3]{\dfrac{27a^6b^2}{64x^3y^3}}$.
26. $\sqrt{\dfrac{49a^3b^2}{36x^2y^6}}$.
27. $\sqrt[3]{\dfrac{512x^2}{125y^3}}$.
28. $2\sqrt[5]{\dfrac{m^6n^3x}{3125}}$.
29. $3\sqrt[4]{\dfrac{x^5y^7}{256}}$.
30. $2\sqrt[3]{\dfrac{(x-y)^3z^3}{(x+y)^3}}$.
31. $\dfrac{4ab}{5c}\sqrt{\dfrac{75c^2d}{16a^2b^2}}$.

Case III.

216. When the Radical Expression is a Fraction, the Denominator of which is not a Perfect Power of the Same Degree as the Radical.

(1) $\sqrt{\frac{5}{8}}=\sqrt{\frac{10}{16}}=\sqrt{10\times\frac{1}{16}}=\frac{1}{4}\sqrt{10}$;

(2) $\sqrt{\frac{7}{12}}=\sqrt{\frac{7}{4\times 3}}=\sqrt{\frac{7\times 3}{4\times 9}}=\sqrt{21\times\frac{1}{36}}=\frac{1}{6}\sqrt{21}$;

(3) $\sqrt[3]{\frac{5}{18}}=\sqrt[3]{\frac{5}{9\times 2}}=\sqrt[3]{\frac{5\times 3\times 4}{27\times 8}}=\sqrt[3]{60\times\frac{1}{27\times 8}}$

$$=\frac{1}{3\times 2}\sqrt[3]{60}=\tfrac{1}{6}\sqrt[3]{60}.$$

We have, therefore, the following rule:

Multiply both terms of the fraction by such a number as will make the denominator a perfect power of the same degree as the radical; and then proceed as in Case II.

Exercise 83.

Simplify:

1. $\sqrt{\frac{1}{2}}$.
2. $\sqrt{\frac{1}{3}}$.
3. $\sqrt{\frac{1}{5}}$.
4. $3\sqrt{\frac{4}{5}}$.
5. $2\sqrt[4]{\frac{5}{32}}$.
6. $3\sqrt{\frac{7}{80}}$.
7. $\sqrt[3]{\frac{5}{3}}$.
8. $\sqrt[3]{\frac{9}{16}}$.
9. $\sqrt[3]{\frac{7}{686}}$.
10. $2\sqrt[3]{\frac{4}{25}}$.
11. $3\sqrt[5]{\frac{5}{81}}$.
12. $2\sqrt[5]{\frac{7}{16}}$.
13. $\sqrt{\frac{a^4c^2}{b^3}}$.
14. $\sqrt[4]{\frac{b^4d^2}{a^3c^2}}$.
15. $\sqrt[3]{\frac{a^2x^4}{b^4}}$.
16. $\sqrt[3]{\frac{7a^2}{125x}}$.
17. $\sqrt{\frac{a^2cy^2}{b^3d^2}}$.
18. $2\sqrt[3]{\frac{3a^3b^2c^2}{4x^2yz^3}}$.

Case IV.

217. To reduce a Mixed Surd to an Entire Surd.

Since $a\sqrt[n]{b} = \sqrt[n]{a^n} \times \sqrt[n]{b} = \sqrt[n]{a^n b}$, we have

(1) $3\sqrt{5} = \sqrt{3^2 \times 5} = \sqrt{9 \times 5} = \sqrt{45}$;

(2) $a^2b\sqrt{bc} = \sqrt{(a^2b)^2 \times bc} = \sqrt{a^4b^2 \times bc} = \sqrt{a^4b^3c}$;

(3) $2x\sqrt[3]{xy} = \sqrt[3]{(2x)^3 \times xy} = \sqrt[3]{8x^3 \times xy} = \sqrt[3]{8x^4y}$;

(4) $3y^2\sqrt[4]{x^3} = \sqrt[4]{(3y^2)^4 \times x^3} = \sqrt[4]{81y^8x^3}$.

We have, therefore, the following rule:

Raise the coefficient to a power of the same degree as the radical, multiply this power by the given surd factor, and indicate the required root of the product.

Exercise 84.

Express as entire surds:

1. $3\sqrt{5}$.	5. $2\sqrt[4]{7}$.	9. $-2\sqrt[3]{y}$.	13. $\frac{9}{11}\sqrt{a}$.
2. $3\sqrt{21}$.	6. $3\sqrt[5]{3}$.	10. $-3\sqrt[5]{y^3}$.	14. $-\frac{1}{2}\sqrt[3]{a^2}$.
3. $3\sqrt[3]{2}$.	7. $2\sqrt[6]{5}$.	11. $-m\sqrt[3]{10}$.	15. $\frac{3}{5}\sqrt{m^3}$.
4. $2\sqrt[3]{5}$.	8. $2\sqrt[4]{2}$.	12. $-2\sqrt[7]{x}$.	16. $-\frac{1}{2}\sqrt[3]{m^7}$.

Case V.

218. To reduce Radicals to a Common Index.

(1) Reduce $\sqrt{2}$ and $\sqrt[3]{3}$ to a common index.

$$\sqrt{2} = 2^{\frac{1}{2}} = 2^{\frac{3}{6}} = \sqrt[6]{2^3} = \sqrt[6]{8}.$$

$$\sqrt[3]{3} = 3^{\frac{1}{3}} = 3^{\frac{2}{6}} = \sqrt[6]{3^2} = \sqrt[6]{9}.$$

We have, therefore, the following rule:

Write the radicals with fractional exponents, and change these fractional exponents to equivalent exponents having the least common denominator. Raise each radical to the power denoted by the numerator, and indicate the root denoted by the common denominator.

Exercise 85.

Reduce to surds of the same order:

1. $\sqrt[4]{3}$ and $\sqrt[6]{2}$.
2. $\sqrt[3]{7}$ and $\sqrt{6}$.
3. $\sqrt{3}$ and $\sqrt[3]{4}$.
4. $\sqrt[4]{a}$ and $\sqrt[3]{b^2}$.
5. $\sqrt{5}$ and $\sqrt[6]{25}$.
6. $3^{\frac{1}{2}}$, $3^{\frac{2}{3}}$, and $3^{\frac{3}{4}}$.
7. $\sqrt[3]{2}$, $\sqrt[4]{3}$, and $\sqrt[6]{5}$.
8. $\sqrt[6]{a^2}$, $\sqrt[3]{b}$, and $\sqrt[7]{c}$.
9. $\sqrt[5]{a^4}$, $\sqrt[10]{c^3}$, and $\sqrt[4]{x^3}$.
10. $\sqrt[4]{x^2y}$, $\sqrt[3]{abc}$, and $\sqrt{2z}$.
11. $\sqrt[5]{x-y}$ and $\sqrt[3]{x+y}$.
12. $\sqrt[3]{a+b}$ and $\sqrt[4]{a-b}$.

NOTE. Surds of different orders may be reduced to surds of the same order and then compared in respect to magnitude.

Arrange in order of magnitude:

13. $2\sqrt[3]{3}$, $3\sqrt{2}$, $\frac{5}{2}\sqrt[4]{4}$.
14. $\sqrt{\frac{3}{5}}$, $\sqrt[3]{\frac{14}{15}}$.
15. $2\sqrt[3]{22}$, $3\sqrt[3]{7}$, $4\sqrt{2}$.
16. $3\sqrt{19}$, $5\sqrt[3]{2}$, $3\sqrt[3]{3}$.

Addition and Subtraction of Radicals.

219. In the addition of surds, each surd must be reduced to its simplest form; and, if the resulting surds are similar,

Find the algebraic sum of the coefficients, and to this sum annex the common surd factor.

If the resulting surds are not similar,

Connect them with their proper signs.

(1) Simplify $\sqrt{27}+\sqrt{48}+\sqrt{147}$.

$$\sqrt{27}=(3^2\times 3)^{\frac{1}{2}}=3\times 3^{\frac{1}{2}}=3\sqrt{3};$$

$$\sqrt{48}=(2^4\times 3)^{\frac{1}{2}}=2^2\times 3^{\frac{1}{2}}=4\times 3^{\frac{1}{2}}=4\sqrt{3};$$

$$\sqrt{147}=(7^2\times 3)^{\frac{1}{2}}=7\times 3^{\frac{1}{2}}=7\sqrt{3}.$$

$$\therefore\ \sqrt{27}+\sqrt{48}+\sqrt{147}=(3+4+7)\sqrt{3}=14\sqrt{3}.$$

(2) Simplify $2\sqrt[3]{320}-3\sqrt[3]{40}$.

$$2\sqrt[3]{320}=2(2^6\times 5)^{\frac{1}{3}}=2\times 2^2\times 5^{\frac{1}{3}}=8\sqrt[3]{5};$$

$$3\sqrt[3]{40}=3(2^3\times 5)^{\frac{1}{3}}=3\times 2\times 5^{\frac{1}{3}}=6\sqrt[3]{5}.$$

$$\therefore\ 2\sqrt[3]{320}-3\sqrt[3]{40}=(8-6)\sqrt[3]{5}=2\sqrt[3]{5}.$$

(3) Simplify $2\sqrt{\frac{5}{3}}-3\sqrt{\frac{3}{5}}+\sqrt{\frac{4}{15}}$,

$$2\sqrt{\tfrac{5}{3}}=2\sqrt{\tfrac{15}{9}}=2\sqrt{15\times\tfrac{1}{9}}=\tfrac{2}{3}\sqrt{15};$$

$$3\sqrt{\tfrac{3}{5}}=3\sqrt{\tfrac{15}{25}}=3\sqrt{15\times\tfrac{1}{25}}=\tfrac{3}{5}\sqrt{15};$$

$$\sqrt{\tfrac{4}{15}}=\sqrt{\frac{4\times 15}{15^2}}=\sqrt{15\times\frac{4}{15^2}}=\tfrac{2}{15}\sqrt{15}.$$

$$\therefore\ 2\sqrt{\tfrac{5}{3}}-3\sqrt{\tfrac{3}{5}}+\sqrt{\tfrac{4}{15}}=(\tfrac{2}{3}-\tfrac{3}{5}+\tfrac{2}{15})\sqrt{15}=\tfrac{1}{5}\sqrt{15}.$$

Exercise 86.

Simplify:

1. $8\sqrt{11}+7\sqrt{11}-10\sqrt{11}$.
2. $3\sqrt{5}-5\sqrt{5}+7\sqrt{5}$.
3. $\sqrt{27}+2\sqrt{48}+3\sqrt{108}$.
4. $\sqrt[3]{128}+\sqrt[3]{686}+\sqrt[3]{16}$.
5. $12\sqrt{72}-3\sqrt{128}$.
6. $2\sqrt{3}+3\sqrt{1\frac{1}{3}}-\sqrt{5\frac{1}{3}}$.
7. $\sqrt{1000}+\sqrt{50}+\sqrt{288}$.
8. $\sqrt[3]{54}+3\sqrt[3]{16}+\sqrt[3]{432}$.
9. $7\sqrt[3]{81}-3\sqrt[3]{1029}$.
10. $\sqrt{\frac{5}{3}}+\sqrt{60}-\sqrt{15}-\sqrt{\frac{3}{5}}$.

11. $\sqrt{\dfrac{a^4c}{b^3}}-\sqrt{\dfrac{a^2c^3}{bd^2}}-\sqrt{\dfrac{a^2cd^2}{bm^2}}.$

12. $\sqrt{\frac{2}{5}}+\sqrt{\frac{1}{10}}-\sqrt{\frac{1}{40}}.$

13. $\sqrt{4a^3b}+\sqrt{25ab^3}-(a-5b)\sqrt{ab}.$

14. $c\sqrt[5]{a^6b^7c^3}-a\sqrt[5]{ab^7c^8}+b\sqrt[5]{a^6b^2c^8}.$

15. $2\sqrt[3]{40}+3\sqrt[3]{108}+\sqrt[3]{500}-\sqrt[3]{320}-2\sqrt[3]{1372}.$

16. $\sqrt{363}-2\sqrt{243}+\sqrt{108}-2\sqrt{147}.$

17. $\sqrt[3]{189}-2\sqrt[3]{448}+\sqrt[3]{875}+\sqrt[3]{1512}.$

18. $\sqrt[4]{162}-\sqrt[4]{512}+2\sqrt[4]{32}-\sqrt[4]{1250}.$

19. $\sqrt[3]{-81}-3\sqrt[3]{-24}+2\sqrt[3]{192}.$

20. $\sqrt{20}+\sqrt{45}-\sqrt{\frac{4}{5}}.$

21. $2\sqrt[3]{a^3b^2}+\sqrt[3]{8b^5}-\dfrac{1}{2}\sqrt[3]{\dfrac{a^6}{b}}.$

22. $\sqrt{50}+\frac{5}{2}\sqrt{288}-\dfrac{1}{\sqrt{2}}-\dfrac{3}{\sqrt{450}}.$

23. $\sqrt{1701}+\frac{1}{4}\sqrt{84}-\frac{1}{5}\sqrt{525}.$

Multiplication of Radicals.

220. Since $\sqrt[n]{a}\times\sqrt[n]{b}=\sqrt[n]{ab}$, we have

(1) $3\sqrt{8}\times 5\sqrt{2}=3\times 5\times\sqrt{8}\times\sqrt{2}=15\sqrt{16}=60;$

(2) $3\sqrt{2}\times 4\sqrt[3]{3}=3\sqrt[6]{8}\times 4\sqrt[6]{9}=12\sqrt[6]{72}.$

We have, therefore, the following rule:

Express the radicals with a common index. Find the product of the coefficients for the required coefficient, and the product of the surd factors for the required surd factor.

Reduce the result to its simplest form.

Exercise 87.

Find the product of:

1. $3\sqrt{2}$ by $4\sqrt{6}$.
2. $2\sqrt{5}$ by $3\sqrt{15}$.
3. $2\sqrt{10}$ by $5\sqrt{14}$.
4. $3\sqrt{27}$ by $7\sqrt{48}$.
5. $2\sqrt[3]{4}$ by $5\sqrt[3]{32}$.
6. $\frac{2}{7}\sqrt{10}$ by $\frac{7}{10}\sqrt{15}$.
7. $5\sqrt{\frac{2}{7}}$ by $\frac{3}{7}\sqrt{162}$.
8. $\frac{3}{8}\sqrt{21}$ by $\frac{9}{10}\sqrt{\frac{7}{20}}$.
9. $\sqrt[3]{108}$ by $5\sqrt{32}$.
10. $5\sqrt[3]{54}$ by $7\sqrt{48}$.

221. Compound radicals are multiplied as follows:

Multiply $2\sqrt{3}+3\sqrt{x}$ by $3\sqrt{3}-4\sqrt{x}$.

$$\begin{array}{l} 2\sqrt{3}+3\sqrt{x} \\ 3\sqrt{3}-4\sqrt{x} \\ \hline 18+9\sqrt{3x} \\ \qquad -8\sqrt{3x}-12x \\ \hline 18+\quad\sqrt{3x}-12x \end{array}$$

Exercise 88.

Find the product of:

1. $(2\sqrt{x}-7)\times 3\sqrt{x}$.
2. $(\sqrt{a}-\sqrt{b})^2$.
3. $(3\sqrt{5}-7\sqrt{2})^2$.
4. $(\sqrt{7}+3\sqrt{3})(\sqrt{7}-2\sqrt{3})$.
5. $(3\sqrt{5}-\sqrt{2})(\sqrt{5}-3\sqrt{2})$.
6. $(\sqrt{2}+\sqrt{3}-\sqrt{5})^2$.
7. $(\sqrt{5}+3\sqrt{2}+\sqrt{7})^2$
8. $(2\sqrt{5}-\sqrt{2}-\sqrt{7})^2$.
9. $(2\sqrt{x}+\sqrt{3-2x})^2$.
10. $(2\sqrt{a^2+b^2}-3\sqrt{a^2-b^2})^2$.

Division of Radicals.

222. Since $\dfrac{\sqrt[n]{ab}}{\sqrt[n]{a}} = \dfrac{\sqrt[n]{a} \times \sqrt[n]{b}}{\sqrt[n]{a}} = \sqrt[n]{b}$, we have

(1) $\dfrac{4\sqrt{8}}{2\sqrt{2}} = 2\sqrt{4} = 4;$

(2) $\dfrac{4\sqrt[3]{3}}{2\sqrt{2}} = \dfrac{4\sqrt[6]{3^2}}{2\sqrt[6]{2^3}} = \dfrac{4\sqrt[6]{3^2 \times 2^3}}{2\sqrt[6]{2^6}} = \sqrt[6]{72}.$

We have, therefore, the following rule:

Express the radicals with a common index. Find the quotient of the coefficients for the required coefficient, and the quotient of the surd factors for the required surd factor.

Reduce the result to its simplest form.

Exercise 89.

Divide:

1. $\sqrt{162}$ by $\sqrt{2}$.
2. $\sqrt[3]{81}$ by $\sqrt[3]{3}$.
3. $\sqrt{2a^{11}}$ by $\sqrt{a^3}$.
4. $\sqrt{\frac{1}{3}}$ by $\sqrt{\frac{3}{7}}$.
5. $\sqrt{\frac{5}{12}}$ by $\sqrt{\frac{3}{5}}$.
6. $2\sqrt{\frac{1}{2}}$ by $\frac{3}{4}\sqrt{\frac{2}{3}}$.
7. $\sqrt{5}$ by $\sqrt[3]{4}$.
8. $\sqrt{\frac{2}{9}}$ by $\sqrt[3]{\frac{1}{2}}$.
9. $\sqrt[3]{\frac{5}{6}}$ by $\sqrt{\frac{3}{5}}$.
10. $3\sqrt{2} + \sqrt{72} - 3\sqrt{8}$ by $\sqrt{3}$.
11. $9\sqrt[3]{2} - 6\sqrt[3]{6} - 3\sqrt[3]{8} + 12\sqrt[3]{32}$ by $3\sqrt[3]{2}$.
12. $\sqrt[3]{2} - \sqrt[3]{6} + \sqrt[3]{10} - \sqrt[3]{12}$ by $\sqrt[3]{2}$.

223. The quotient of one surd by another may be found by *rationalizing the divisor;* that is, by multiplying the dividend and divisor by a factor which will free the divisor of surds.

224. This method is of great utility when we wish to find the approximate numerical value of the quotient of two simple surds; and is the method *required* when the divisor is a compound surd.

(1) Given $\sqrt{2} = 1.41421$, find the value of $\dfrac{5}{\sqrt{2}}$.

$$\frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{5\sqrt{2}}{2} = \frac{7.07105}{2} = 3.53553.$$

(2) Divide $3\sqrt{5} - 4\sqrt{2}$ by $2\sqrt{5} + 3\sqrt{2}$.

$$\frac{3\sqrt{5} - 4\sqrt{2}}{2\sqrt{5} + 3\sqrt{2}} = \frac{(3\sqrt{5} - 4\sqrt{2})(2\sqrt{5} - 3\sqrt{2})}{(2\sqrt{5} + 3\sqrt{2})(2\sqrt{5} - 3\sqrt{2})} = \frac{54 - 17\sqrt{10}}{20 - 18}$$

$$= \frac{54 - 17\sqrt{10}}{2} = 27 - 8\tfrac{1}{2}\sqrt{10}.$$

By two operations the divisor may be rationalized when it consists of *three* quadratic surds.

Thus, if $\sqrt{6} + \sqrt{3} - \sqrt{2}$ be multiplied by $\sqrt{6} - \sqrt{3} + \sqrt{2}$, the result will be $6 - 5 + 2\sqrt{6} = 1 + 2\sqrt{6}$; and if $1 + 2\sqrt{6}$ be multiplied by $1 - 2\sqrt{6}$, the product will be $1 - 24 = -23$.

Exercise 90.

Find the approximate value of:

1. $\dfrac{2}{\sqrt{3}}$. **2.** $\dfrac{1}{\sqrt{5}}$. **3.** $\dfrac{7\sqrt{2}}{\sqrt{3}}$. **4.** $\dfrac{2\sqrt{5}}{3\sqrt{2}}$.

Divide:

5. 3 by $\sqrt{7} + \sqrt{5}$.

6. 7 by $2\sqrt{5} - \sqrt{6}$.

7. 6 by $5 - 2\sqrt{6}$.

8. $4 - \sqrt{2}$ by $1 + \sqrt{2}$.

9. $\sqrt{5} + \sqrt{2}$ by $\sqrt{5} - \sqrt{2}$.

10. $7 + 2\sqrt{10}$ by $7 - 2\sqrt{10}$.

11. $\sqrt{5} - \sqrt{6}$ by $2\sqrt{5} - \sqrt{6}$.

12. $a + b$ by $a - \sqrt{b}$.

13. 1 by $\sqrt{5} + \sqrt{3} + \sqrt{7}$.

14. 2 by $\sqrt{5} - 3\sqrt{2} + \sqrt{7}$.

Involution and Evolution of Radicals.

225. Any power or root of a radical is easily found by using fractional exponents.

(1) Find the square of $2\sqrt[3]{a}$.

$$(2\sqrt[3]{a})^2 = (2a^{\frac{1}{3}})^2 = 2^2a^{\frac{2}{3}} = 4a^{\frac{2}{3}} = 4\sqrt[3]{a^2}.$$

(2) Find the cube of $2\sqrt{a}$.

$$(2\sqrt{a})^3 = (2a^{\frac{1}{2}})^3 = 2^3a^{\frac{3}{2}} = 8a^{\frac{3}{2}} = 8a\sqrt{a}.$$

(3) Find the square root of $4x\sqrt{a^3b^3}$.

$$(4x\sqrt{a^3b^3})^{\frac{1}{2}} = (4xa^{\frac{3}{2}}b^{\frac{3}{2}})^{\frac{1}{2}} = 4^{\frac{1}{2}}x^{\frac{1}{2}}a^{\frac{3}{4}}b^{\frac{3}{4}} = 4^{\frac{1}{2}}x^{\frac{2}{4}}a^{\frac{3}{4}}b^{\frac{3}{4}} = 2\sqrt[4]{a^3b^3x^2}.$$

(4) Find the cube root of $4x\sqrt{a^3b^3}$.

$$(4x\sqrt{a^3b^3})^{\frac{1}{3}} = (4xa^{\frac{3}{2}}b^{\frac{3}{2}})^{\frac{1}{3}} = 4^{\frac{1}{3}}x^{\frac{1}{3}}a^{\frac{1}{2}}b^{\frac{1}{2}} = 4^{\frac{2}{6}}x^{\frac{2}{6}}a^{\frac{3}{6}}b^{\frac{3}{6}} = \sqrt[6]{16a^3b^3x^2}.$$

Exercise 91.

Perform the operations indicated:

1. $(\sqrt[9]{8})^4$.	**4.** $(a\sqrt[3]{a})^3$.	**7.** $(\sqrt[3]{256})^{\frac{1}{4}}$.
2. $(\sqrt[8]{64})^3$.	**5.** $(x\sqrt[3]{x})^2$.	**8.** $\left(\frac{a}{3}\sqrt{\frac{a}{3}}\right)^{\frac{1}{3}}$.
3. $(\sqrt[3]{4})^2$.	**6.** $(3\sqrt[6]{3})^2$.	**9.** $(2\sqrt[6]{4a^4b})^{\frac{1}{3}}$.

Properties of Quadratic Surds.

226. *The product or quotient of two dissimilar quadratic surds is a quadratic surd.* Thus,

$$\sqrt{ab}\times\sqrt{abc} = ab\sqrt{c}; \qquad \sqrt{abc}\div\sqrt{ab} = \sqrt{c}.$$

For every quadratic surd, when simplified, will have under the radical sign one or more factors raised only to the first power; and two surds which are *dissimilar* cannot have *all* these factors alike.

Hence, their product or quotient will have *at least one factor* raised only to the *first* power, and will therefore be a surd.

227. *The sum or difference of two dissimilar quadratic surds cannot be a rational number, nor can it be expressed as a single surd.*

For if $\sqrt{a} \pm \sqrt{b}$ could equal a rational number c, we should have, by squaring,

$$a \pm 2\sqrt{ab} + b = c^2;$$

that is,

$$\pm 2\sqrt{ab} = c^2 - a - b.$$

Now, as the right side of this equation is rational, the left side is rational; but, by § 226, $\sqrt{ab}$ cannot be rational. Therefore, $\sqrt{a} \pm \sqrt{b}$ cannot be rational.

In like manner, it may be shown that $\sqrt{a} \pm \sqrt{b}$ cannot be expressed as a single surd $\sqrt{c}$.

228. *A quadratic surd cannot equal the sum of a rational number and a surd.*

For if $\sqrt{a}$ could equal $c + \sqrt{b}$, we should have, by squaring,

$$a = c^2 + 2c\sqrt{b} + b,$$

and, by transposing,

$$2c\sqrt{b} = a - b - c^2.$$

That is, a surd equal to a rational number, which is impossible.

229. *If $a + \sqrt{b} = x + \sqrt{y}$, then a will equal x, and b will equal y.*

For, by transposing, $\sqrt{b} - \sqrt{y} = x - a$; and if b were not equal to y, the difference of two unequal surds would be rational, which by § 227 is impossible.

$$\therefore\ b = y, \text{ and } a = x.$$

In like manner, if $a - \sqrt{b} = x - \sqrt{y}$, a will equal x, and b will equal y.

230. To extract the Square Root of a Binomial Surd.

Extract the square root of $a+\sqrt{b}$.

Suppose $$\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}. \qquad (1)$$

By squaring, $$a+\sqrt{b}=x+2\sqrt{xy}+y. \qquad (2)$$

$$\therefore\ a=x+y \text{ and } \sqrt{b}=2\sqrt{xy}. \qquad \S\ 229$$

Therefore, $$a-\sqrt{b}=x-2\sqrt{xy}+y, \qquad (3)$$

and $$\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}. \qquad (4)$$

Multiplying (1) by (4),

$$\sqrt{a^2-b}=x-y.$$

But $$a=x+y.$$

Adding, and dividing by 2, $$x=\frac{a+\sqrt{a^2-b}}{2}.$$

Subtracting, and dividing by 2,

$$y=\frac{a-\sqrt{a^2-b}}{2}.$$

$$\therefore\ \sqrt{a+\sqrt{b}}=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}+\sqrt{\frac{a-\sqrt{a^2-b}}{2}}.$$

From these two values of x and y, it is evident that this method is practicable only when a^2-b is a perfect square.

(1) Extract the square root of $7+4\sqrt{3}$.

Let $$\sqrt{x}+\sqrt{y}=\sqrt{7+4\sqrt{3}}.$$

Then $$\sqrt{x}-\sqrt{y}=\sqrt{7-4\sqrt{3}}.$$

Multiplying, $$x-y=\sqrt{49-48}.$$

$$\therefore\ x-y=1.$$

But $$x+y=7.$$

$$\therefore\ x=4, \text{ and } y=3.$$

$$\therefore\ \sqrt{x}+\sqrt{y}=2+\sqrt{3}.$$

$$\therefore\ \sqrt{7+4\sqrt{3}}=2+\sqrt{3}.$$

A root may often be obtained by inspection. For this purpose, write the given expression in the form $a + 2\sqrt{b}$, and determine two numbers whose sum is equal to a, and whose product is equal to b.

(2) Find by inspection the square root of $18 + 2\sqrt{77}$.

It is required to find two numbers whose sum is 18 and whose product is 77; and these are evidently 11 and 7.

Then $\quad 18 + 2\sqrt{77} = 11 + 7 + 2\sqrt{11 \times 7},$

$$= (\sqrt{11} + \sqrt{7})^2.$$

That is, $\sqrt{11} + \sqrt{7} =$ square root of $18 + 2\sqrt{77}$.

(3) Find by inspection the square root of $75 - 12\sqrt{21}$.

It is necessary that the coefficient of the surd be 2; therefore, $75 - 12\sqrt{21}$ must be put in the form

$$75 - 2\sqrt{756}.$$

The two numbers whose sum is 75 and whose product is 756 are 63 and 12.

Then $\quad 75 - 2\sqrt{756} = 63 + 12 - 2\sqrt{63 \times 12},$

$$= (\sqrt{63} - \sqrt{12})^2 = (3\sqrt{7} - 2\sqrt{3})^2.$$

That is, $3\sqrt{7} - 2\sqrt{3} =$ square root of $75 - 12\sqrt{21}$.

Exercise 92.

Extract the square roots of:

1. $14 + 6\sqrt{5}$.
2. $17 + 4\sqrt{15}$.
3. $10 + 2\sqrt{21}$.
4. $16 + 2\sqrt{55}$.
5. $9 - 2\sqrt{14}$.
6. $20 - 8\sqrt{6}$.
7. $9 - 6\sqrt{2}$.
8. $94 - 42\sqrt{5}$.
9. $13 - 2\sqrt{30}$.
10. $11 - 6\sqrt{2}$.
11. $14 - 4\sqrt{6}$.
12. $38 - 12\sqrt{10}$.
13. $103 - 12\sqrt{11}$.
14. $57 - 12\sqrt{15}$.
15. $3\frac{1}{2} - \sqrt{10}$.
16. $2a + 2\sqrt{a^2 - b^2}$.
17. $a^2 - 2b\sqrt{a^2 - b^2}$.
18. $87 - 12\sqrt{42}$.
19. $(a+b)^2 - 4(a-b)\sqrt{ab}$.

CHAPTER XVIII.

IMAGINARY EXPRESSIONS.

231. An **imaginary expression** is any expression which involves the indicated even root of a negative number.

It will be shown hereafter that *any* indicated even root of a negative number may be made to assume a form which involves only an indicated *square root* of a negative number. In considering imaginary expressions, we accordingly need consider only expressions which involve the indicated square roots of negative numbers.

Imaginary expressions are also called **imaginary numbers** and **complex numbers.** In distinction from imaginary numbers, all other numbers are called **real numbers.**

232. Imaginary Square Roots. If a and b are both positive, we have

$$\text{I. } \sqrt{ab} = \sqrt{a} \times \sqrt{b}. \qquad \text{II. } (\sqrt{a})^2 = a.$$

If one of the two numbers a and b is positive and the other negative, Law I. is *assumed* still to apply; we have, accordingly:

$$\sqrt{-4} = \sqrt{4(-1)} = \sqrt{4} \times \sqrt{-1} = 2\sqrt{-1};$$
$$\sqrt{-5} = \sqrt{5(-1)} = \sqrt{5} \times \sqrt{-1} = 5^{\frac{1}{2}}\sqrt{-1};$$
$$\sqrt{-a} = \sqrt{a(-1)} = \sqrt{a} \times \sqrt{-1} = a^{\frac{1}{2}}\sqrt{-1};$$

and so on.

It appears, then, that every imaginary square root can be made to assume the form $a\sqrt{-1}$, where a is a real number.

233. The symbol $\sqrt{-1}$ is called the **imaginary unit,** and may be defined as an expression the square of which is -1.

Hence, $\sqrt{-1}\times\sqrt{-1}=(\sqrt{-1})^2=-1$;

$$\begin{aligned}\sqrt{-a}\times\sqrt{-b}&=\sqrt{a}\times\sqrt{-1}\times\sqrt{b}\times\sqrt{-1}\\&=\sqrt{a}\times\sqrt{b}\times(\sqrt{-1})^2\\&=\sqrt{ab}\times(-1)\\&=-\sqrt{ab}.\end{aligned}$$

234. It will be useful to form the successive powers of the imaginary unit.

$$\begin{aligned}&(\sqrt{-1}) \quad \dots\dots\dots\dots &&=+\sqrt{-1};\\&(\sqrt{-1})^2 \quad \dots\dots\dots\dots &&=-1;\\&(\sqrt{-1})^3=(\sqrt{-1})^2\sqrt{-1} &&=(-1)\sqrt{-1}=-\sqrt{-1};\\&(\sqrt{-1})^4=(\sqrt{-1})^2(\sqrt{-1})^2=(-1)(-1) &&=+1;\\&(\sqrt{-1})^5=(\sqrt{-1})^4\sqrt{-1} &&=(+1)\sqrt{-1}=+\sqrt{-1};\end{aligned}$$

and so on. We have, therefore, if n is any integer,

$$\begin{aligned}(\sqrt{-1})^{4n+1}&=+\sqrt{-1};\\(\sqrt{-1})^{4n+2}&=-1;\\(\sqrt{-1})^{4n+3}&=-\sqrt{-1};\\(\sqrt{-1})^{4n+4}&=+1.\end{aligned}$$

235. Every imaginary expression may be made to assume the form $a+b\sqrt{-1}$, where a and b are real numbers, and may be integers, fractions, or surds.

If $b=0$, the expression consists of only the real part a, and is therefore real.

If $a=0$, the expression consists of only the imaginary part $b\sqrt{-1}$, and is called a **pure imaginary.**

236. The form $a+b\sqrt{-1}$ is the **typical form** of imaginary expressions.

Reduce to the typical form $6+\sqrt{-8}$.

This may be written $6+\sqrt{8}\times\sqrt{-1}$, or $6+2\sqrt{2}\times\sqrt{-1}$; here $a=6$, and $b=2\sqrt{2}$.

237. Two expressions of the form $a+b\sqrt{-1}$, $a-b\sqrt{-1}$, are called **conjugate imaginaries.**

To find the sum and product of two conjugate imaginaries,

$$\begin{array}{ll} & a+b\sqrt{-1} \\ & a-b\sqrt{-1} \\ \hline \text{The sum is} & 2a \end{array}$$

$$\begin{array}{ll} & a+b\sqrt{-1} \\ & a-b\sqrt{-1} \\ \hline & a^2+ab\sqrt{-1} \\ & \qquad -ab\sqrt{-1}+b^2 \\ \hline \text{The product is} & a^2 \qquad\qquad\quad +b^2 \end{array}$$

From the above it appears that the *sum* and *product* of two conjugate imaginaries are both *real.*

238. *An imaginary expression cannot be equal to a real number.*

For, if possible, let

$$a+b\sqrt{-1}=c.$$

Then transposing a, $\quad b\sqrt{-1}=c-a,$
and squaring, $\quad -b^2=(c-a)^2.$

Since b^2 and $(c-a)^2$ are both positive, we have a negative number equal to a positive number, which is impossible.

239. *If two imaginary expressions are equal, the real parts are equal and the imaginary parts are equal.*

For let $$a+b\sqrt{-1}=c+d\sqrt{-1}.$$

Then $$(b-d)\sqrt{-1}=c-a;$$

squaring, $$-(b-d)^2=(c-a)^2,$$

which is impossible unless $b=d$ and $a=c$.

240. *If x and y are real and $x+y\sqrt{-1}=0$, then $x=0$ and $y=0$.*

For, $$y\sqrt{-1}=-x,$$
$$-y^2=x^2,$$
$$x^2+y^2=0,$$

which is true only when $x=0$ and $y=0$.

241. Operations with Imaginaries.

(1) Add $5+7\sqrt{-1}$ and $8-9\sqrt{-1}$.

The sum is $$5+8+7\sqrt{-1}-9\sqrt{-1},$$

or $$13-2\sqrt{-1}.$$

(2) Multiply $3+2\sqrt{-1}$ by $5-4\sqrt{-1}$.

$$\begin{aligned}(3+2\sqrt{-1})(5-4\sqrt{-1})&\\ &=15-12\sqrt{-1}+10\sqrt{-1}-8(-1)\\ &=23-2\sqrt{-1}.\end{aligned}$$

(3) Divide $14+5\sqrt{-1}$ by $2-3\sqrt{-1}$.

$$\begin{aligned}\frac{14+5\sqrt{-1}}{2-3\sqrt{-1}}&=\frac{(14+5\sqrt{-1})(2+3\sqrt{-1})}{(2-3\sqrt{-1})(2+3\sqrt{-1})}\\ &=\frac{13+52\sqrt{-1}}{4-(-9)}\\ &=\frac{13+52\sqrt{-1}}{13}\\ &=1+4\sqrt{-1}.\end{aligned}$$

Exercise 93.

Reduce to the form $b\sqrt{-1}$ and add:

1. $\sqrt{-4}+\sqrt{-25}$.
2. $\sqrt{-81}-\sqrt{-36}$.
3. $\sqrt{-144}+\sqrt{-100}$.
4. $\sqrt{-256}-\sqrt{-16}$.
5. $\sqrt{-121}-\sqrt{-49}$.
6. $\sqrt{-a^4}+\sqrt{-4a^2}-\sqrt{-16a^4}$.
7. $\sqrt{-16a^6}+\sqrt{-49a^2}+\sqrt{-4a^4}$.
8. $\sqrt{-m}+\sqrt{-n}-\sqrt{-4}$.
9. $3a\sqrt{-4a^2}+2a^2\sqrt{-49}$.
10. $\sqrt{18}+\sqrt{-18}-\sqrt{-8}$.

Reduce to the form $b\sqrt{-1}$ and multiply:

11. $1+\sqrt{-4}$ by $1-\sqrt{-4}$.
12. $4+\sqrt{-3}$ by $4-\sqrt{-3}$.
13. $\sqrt{3}-2\sqrt{-2}$ by $\sqrt{3}+2\sqrt{-2}$.
14. $\sqrt{54}-\sqrt{-2}$ by $\sqrt{54}+\sqrt{-2}$.
15. $\sqrt{-a}+\sqrt{-b}$ by $\sqrt{-a}-\sqrt{-b}$.
16. $a\sqrt{-a^2b^4}$ by $a\sqrt{-a^4b^5}$.
17. $2\sqrt{3}-\sqrt{-5}$ by $2\sqrt{3}+\sqrt{-5}$.
18. $\sqrt{-10}$ by $\sqrt{-2}$.

Reduce to the form $b\sqrt{-1}$ and divide:

19. $\sqrt{-12}$ by $\sqrt{-3}$.
20. $\sqrt{15}$ by $\sqrt{-5}$.
21. $\sqrt{-5}$ by $\sqrt{-20}$.
22. a by $\sqrt{-a}$.
23. $-\sqrt{25}$ by $\sqrt{-5}$.
24. $-\sqrt{-25}$ by $-\sqrt{-5}$.
25. $4\sqrt{-20}$ by $-2\sqrt{-25}$.
26. $4+\sqrt{-2}$ by $2-\sqrt{-2}$.

CHAPTER XIX.

QUADRATIC EQUATIONS.

242. We have already considered equations of the first degree in one or more unknowns. We now proceed to the treatment of equations containing one or more unknowns to a degree not exceeding the second. An equation which contains the *square* of the unknown, but no higher power, is called a **quadratic equation.**

243. A quadratic equation which involves but one unknown number can contain only:

(1) Terms involving the square of the unknown number.

(2) Terms involving the first power of the unknown number.

(3) Terms which do not involve the unknown number.

Collecting similar terms, every quadratic equation can be made to assume the form

$$ax^2 + bx + c = 0,$$

where a, b, and c are known numbers, and x the unknown number.

If a, b, c are numbers expressed by figures, the equation is a **numerical quadratic.** If a, b, c are numbers represented wholly or in part by letters, the equation is a **literal quadratic.**

244. In the equation $ax^2 + bx + c = 0$, a, b, and c are called the **coefficients** of the equation. The third term c is called the **constant term.**

If the first power of x is wanting, the equation is a **pure quadratic**; in this case $b = 0$.

If the first power of x is present, the equation is an **affected** or **complete quadratic.**

PURE QUADRATIC EQUATIONS.

245. Examples.

(1) Solve the equation $5x^2 - 48 = 2x^2$.

We have	$5x^2 - 48 = 2x^2.$
Collect the terms,	$3x^2 = 48.$
Divide by 3,	$x^2 = 16.$
Extract the square root,	$x = \pm 4.$

It will be observed that there are *two* roots, and that these are numerically equal, but of opposite signs. There can be only two roots, since any number has only two square roots.

It may seem as though we ought to write the sign $\pm$ before the x as well as before the 4. If we do this, we have $+x = +4$, $-x = -4$, $+x = -4$, $-x = +4$.

From the first and second equations, $x = 4$; from the third and fourth, $x = -4$; these values of x are both given by the equation $x = \pm 4$. Hence it is *unnecessary* to write the $\pm$ sign on *both* sides of the reduced equation.

(2) Solve the equation $3x^2 - 15 = 0$.

We have	$3x^2 = 15,$
or	$x^2 = 5.$
Extract the square root,	$x = \pm\sqrt{5}.$

The roots cannot be found exactly, since the square root of 5 cannot be found exactly; they can, however, be determined approximately to any required degree of accuracy; for example, the positive square root of 5 lies between 2.23606 and 2.23607.

(3) Solve the equation $3x^2 + 15 = 0$.

We have	$3x^2 = -15,$
or	$x^2 = -5.$
Extract the square root,	$x = \pm\sqrt{-5}.$

There is no square root of a negative number, since the square of any number, positive or negative, is necessarily positive.

The square root of -5 differs from the square root of $+5$ in that the latter can be found as accurately as we please, while the former cannot be found at all.

246. A root which can be found exactly is called an **exact** or **rational** root. Such roots are either whole numbers or fractions.

A root which is indicated but can be found only approximately is called a **surd.** Such roots involve the roots of imperfect powers.

Rational and surd roots are together called **real** roots.

A root which is indicated but cannot be found, either exactly or approximately, is called an **imaginary** root. Such roots involve the even roots of negative numbers.

Exercise 94.

Solve:

1. $x^2-3=46.$

2. $2(x^2-1)-3(x^2+1)+14=0.$

3. $\dfrac{x^2-5}{3}+\dfrac{2x^2+1}{6}=\dfrac{1}{2}.$

4. $\dfrac{3}{1+x}+\dfrac{3}{1-x}=8.$

5. $\dfrac{3}{4x^2}-\dfrac{1}{6x^2}=\dfrac{7}{3}.$

6. $5x^2-9=2x^2+24.$

7. $(x+2)^2=4x+5.$

8. $\dfrac{x^2}{5}-\dfrac{x^2-10}{15}=7-\dfrac{50+x^2}{25}.$

9. $\dfrac{3x^2-27}{x^2+3}+\dfrac{90+4x^2}{x^2+9}=7.$

10. $8x+\dfrac{7}{x}=\dfrac{65x}{7}.$

11. $\dfrac{4x^2+5}{10}-\dfrac{2x^2-5}{15}=\dfrac{7x^2-25}{20}.$

12. $\dfrac{10x^2+17}{18}-\dfrac{12x^2+2}{11x^2-8}=\dfrac{5x^2-4}{9}.$

13. $\frac{14x^2+16}{21} - \frac{2x^2+8}{8x^2-11} = \frac{2x^2}{3}$.

14. $x^2 + bx + a = bx(1 - bx)$.

15. $mx^2 + n = q$.

16. $x^2 - ax + b = ax(x-1)$.

Affected Quadratic Equations.

247. Since $(x \pm b)^2 = x^2 \pm 2bx + b^2$, it is evident that the expression $x^2 \pm 2bx$ lacks only the *third term*, b^2, of being a perfect square.

This third term is the square of half the coefficient of x.

Every affected quadratic may be made to assume the form $x^2 \pm 2bx = c$, by dividing the equation through by the coefficient of x^2.

To **solve** such an equation:

The first step is to add to both members *the square of half the coefficient of x.* This is called completing the square.

The second step is to *extract the square root* of each member of the resulting equation.

The third step is to *reduce* the two resulting simple equations.

(1) Solve the equation $x^2 - 8x = 20$.

We have $x^2 - 8x = 20.$
Complete the square, $x^2 - 8x + 16 = 36.$
Extract the square root, $x - 4 = \pm 6.$
Reduce, $x = 4 + 6 = 10,$
or $x = 4 - 6 = -2.$

The roots are 10 and -2.

Verify by putting these numbers for x in the given equation:

$x = 10,$	$x = -2,$
$10^2 - 8(10) = 20,$	$(-2)^2 - 8(-2) = 20,$
$100 - 80 = 20.$	$4 + 16 = 20.$

(2) Solve the equation $\dfrac{x+1}{x-1}=\dfrac{4x-3}{x+9}$.

Free from fractions, $(x+1)(x+9)=(x-1)(4x-3)$.

Simplify, $3x^2-17x=6$.

We can reduce the equation to the form x^2-2bx by dividing by 3.

Divide by 3, $x^2-\frac{17}{3}x=2$.

Half the coefficient of x is $\frac{1}{2}$ of $-\frac{17}{3}=-\frac{17}{6}$, and the square of $-\frac{17}{6}$ is $\frac{289}{36}$. Add the square of $-\frac{17}{6}$ to both sides, and we have

$$x^2-\frac{17x}{3}+\left(\frac{17}{6}\right)^2=2+\frac{289}{36},$$

or

$$x^2-\tfrac{17}{3}x+\left(\frac{17}{6}\right)^2=\frac{361}{36}.$$

Extract the square root, $x-\dfrac{17}{6}=\pm\dfrac{19}{6}$.

$$\therefore\ x=\frac{17}{6}+\frac{19}{6}=\frac{36}{6}=6,$$

or

$$x=\frac{17}{6}-\frac{19}{6}=-\frac{2}{6}=-\frac{1}{3}.$$

The roots are 6 and $-\dfrac{1}{3}$.

Verify by putting these numbers for x in the original equation:

$$x=6,$$

$$\frac{6+1}{6-1}=\frac{24-3}{6+9},$$

$$\frac{7}{5}=\frac{21}{15}.$$

That is, $\dfrac{7}{5}=\dfrac{7}{5}$.

$$x=-\frac{1}{3},$$

$$\frac{-\frac{1}{3}+1}{-\frac{1}{3}-1}=\frac{-\frac{4}{3}-3}{-\frac{1}{3}+9},$$

$$-\frac{2}{4}=-\frac{13}{26}.$$

That is, $-\dfrac{1}{2}=-\dfrac{1}{2}$.

248. When the coefficient of x^2 is not unity, we may proceed as in the preceding section, or we may complete the square by another method.

Since $(ax \pm b)^2$ is identical with $a^2x^2 \pm 2abx + b^2$, it is evident that the expression $a^2x^2 \pm 2abx$ lacks only the third term, b^2, of being a perfect square.

This third term is the square of the quotient obtained by dividing the second term by twice the square root of the first term.

Every affected quadratic may be made to assume the form $a^2x^2 \pm 2abx = c$ (§ 247).

To **solve** such an equation:

The first step is to *complete the square;* to do this, *we divide the second term by twice the square root of the first term, square the quotient, and add the result to both members of the equation.*

The second step is to *extract the square root* of each member of the resulting equation.

The third step is to *reduce* the two resulting simple equations.

249. Numerical Quadratics are solved as follows:

(1) Solve the equation $16x^2 + 5x - 3 = 7x^2 - x + 45$.

$$16x^2 + 5x - 3 = 7x^2 - x + 45.$$

Simplify, $\quad 9x^2 + 6x = 48.$

Complete the square, $\quad 9x^2 + 6x + 1 = 49.$

Extract the square root, $\quad 3x + 1 = \pm 7.$

Reduce, $\quad 3x = -1 + 7 \text{ or } -1 - 7;$

$$3x = 6 \text{ or } -8.$$

$$\therefore x = 2 \text{ or } -2\tfrac{2}{3}.$$

Verify by substituting 2 for x in the equation

$$16x^2 + 5x - 3 = 7x^2 - x + 45.$$

$$16(2)^2 + 5(2) - 3 = 7(2)^2 - (2) + 45,$$

$$64 + 10 - 3 = 28 - 2 + 45,$$

$$71 = 71.$$

Verify by substituting $-2\frac{2}{3}$ for x in the equation

$$16x^2+5x-3=7x^2-x+45.$$

$$16\left(-\frac{8}{3}\right)^2+5\left(-\frac{8}{3}\right)-3=7\left(-\frac{8}{3}\right)^2-\left(-\frac{8}{3}\right)+45,$$

$$\frac{1024}{9}-\frac{40}{3}-3=\frac{448}{9}+\frac{8}{3}+45,$$

$$1024-120-27=448+24+405,$$

$$877=877.$$

(2) Solve the equation $3x^2-4x=32.$

Since the exact root of 3, the coefficient of x^2, cannot be found, it is necessary to multiply or divide each term of the equation by 3 to make the coefficient of x^2 a *square number.*

Multiply by 3, $\quad 9x^2-12x=96.$

Complete the square, $\quad 9x^2-12x+4=100.$

Extract the square root, $\quad 3x-2=\pm 10.$

Reduce, $\quad 3x=2+10 \text{ or } 2-10;$

$$3x=12 \text{ or } -8.$$

$$\therefore\ x=4 \text{ or } -2\tfrac{2}{3}.$$

Or, divide by 3, $\quad x^2-\dfrac{4x}{3}=\dfrac{32}{3}.$

Complete the square, $\quad x^2-\dfrac{4x}{3}+\dfrac{4}{9}=\dfrac{32}{3}+\dfrac{4}{9}=\dfrac{100}{9}.$

Extract the square root, $\quad x-\dfrac{2}{3}=\pm\dfrac{10}{3}.$

$$\therefore\ x=\frac{2\pm 10}{3}$$

$$=4 \text{ or } -2\tfrac{2}{3}.$$

Verify by substituting 4 for x in the original equation,

$$48-16=32,$$

$$32=32.$$

Verify by substituting $-2\frac{2}{3}$ for x in the original equation,

$$21\tfrac{1}{3}+(10\tfrac{2}{3})=32,$$

$$32=32.$$

(3) Solve the equation $-3x^2+5x=-2$.

Since the *even* root of a *negative* number is impossible, it is necessary to change the sign of each term. The resulting equation is

$$3x^2-5x=2.$$

Multiply by 3, $$9x^2-15x=6.$$

Complete the square, $$9x^2-15x+\frac{25}{4}=\frac{49}{4}.$$

Extract the square root, $$3x-\frac{5}{2}=\pm\frac{7}{2}.$$

Reduce, $$3x=\frac{5\pm 7}{2};$$

$$3x=6 \text{ or } -1.$$

$$\therefore\ x=2 \text{ or } -\frac{1}{3}.$$

Or, divide by 3, $$x^2-\frac{5x}{3}=\frac{2}{3}.$$

Complete the square, $$x^2-\frac{5x}{3}+\frac{25}{36}=\frac{49}{36}.$$

Extract the square root, $$x-\frac{5}{6}=\pm\frac{7}{6}.$$

$$\therefore\ x=\frac{5\pm 7}{6},$$

$$=2 \text{ or } -\frac{1}{3}.$$

If the equation $3x^2-5x=2$ is multiplied by *four times the coefficient of* x^2, fractions will be avoided:

$$36x^2-60x=24.$$

Complete the square, $$36x^2-60x+25=49.$$

Extract the square root, $$6x-5=\pm 7.$$

$$6x=5\pm 7.$$

$$\therefore\ x=2 \text{ or } -\frac{1}{3}.$$

The number added to complete the square by this last method is *the square of the coefficient of* x in the original equation $3x^2-5x=2$.

NOTE. If the coefficient of x is an *even* number, we may multiply by the *coefficient* of x^2, and add to each member the square of *half* the coefficient of x in the given equation.

(4) Solve the equation $\dfrac{3}{5-x}-\dfrac{1}{2x-5}=2.$

Simplify, $4x^2-23x=-30.$

Multiply by four times the coefficient of x^2, and add to each side the square of the coefficient of x,

$$64x^2-(\ )+(23)^2=529-480=49.$$

Extract the root, $8x-23=\pm 7.$

Reduce, $8x=23\pm 7;$

$$8x=30 \text{ or } 16.$$

$$\therefore\ x=3\tfrac{3}{4} \text{ or } 2.$$

If a trinomial is a perfect square, its root is found by taking the roots of the *first* and *third* terms and connecting them by the *sign* of the middle term. It is not necessary, therefore, in completing the square, to write the middle term, but its place may be indicated as in this example.

(5) Solve the equation $72x^2-30x=-7.$

Since $72=2^3\times 3^2$, if the equation is multiplied by 2, the coefficient of x^2 in the resulting equation, $144x^2-60x=-14$, will be a square number, and the term required to complete the square will be $(\frac{60}{24})^2=(\frac{5}{2})^2=\frac{25}{4}$. Hence, if the original equation is multiplied by 4×2, the coefficient of x^2 in the result will be a square number, and fractions will be avoided in the work.

Multiply the given equation by 8,

$$576x^2-240x=-56.$$

Complete the square,

$$576x^2-(\ )+25=-31.$$

Extract the root, $24x-5=\pm\sqrt{-31}.$

Reduce, $24x=5\pm\sqrt{-31}.$

$$\therefore\ x=\tfrac{1}{24}(5\pm\sqrt{-31}).$$

Exercise 95.

Solve:

1. $x^2+4x=12.$
2. $x^2-6x=16.$
3. $x^2-12x+6=\frac{1}{4}.$
4. $x^2-7x=8.$
5. $3x^2-4x=7.$
6. $12x^2+x-1=0.$
7. $x^2-x=6.$
8. $5x^2-3x=2.$
9. $2x^2-27x=14.$

10. $x^2 - \frac{2x}{3} + \frac{1}{12} = 0.$

11. $\frac{x^2}{2} - \frac{x}{3} = 2(x+2).$

12. $\frac{3x}{4} + \frac{4}{3x} = \frac{13}{6}.$

13. $\frac{x+1}{x+4} = \frac{2x-1}{x+6}.$

14. $\frac{x}{x+1} - \frac{x+3}{2(x+4)} = -\frac{1}{18}.$

15. $\frac{2}{x-1} = \frac{3}{x-2} + \frac{2}{x-4}.$

16. $5x(x-3) - 2(x^2-6) = (x+3)(x+4).$

17. $\frac{3x}{2(x+1)} - \frac{5}{8} = \frac{3x^2}{x^2-1} - \frac{23}{4(x-1)}.$

18. $(x-2)(x-4) - 2(x-1)(x-3) = 0.$

19. $\frac{1}{7}(x-4) - \frac{2}{5}(x-2) = \frac{1}{x}(2x+3).$

20. $\frac{2}{5}(3x^2 - x - 5) - \frac{1}{3}(x^2-1) = 2(x-2)^2.$

21. $\frac{2x}{15} + \frac{3x-50}{3(10+x)} = \frac{12x+70}{190}.$

22. $\frac{x}{x^2-1} = \frac{15-7x}{8(1-x)}.$

23. $\frac{2x-1}{x-1} + \frac{1}{6} = \frac{2x-3}{x-2}.$

24. $\frac{x+2}{x-1} - \frac{4-x}{2x} = \frac{7}{3}.$

25. $x - \frac{14x-9}{8x-3} = \frac{x^2-3}{x+1}.$

26. $1 - \frac{x+5}{2x+1} = \frac{x-6}{x-2}.$

27. $\frac{x}{7-x} + \frac{7-x}{x} = 2\frac{9}{10}.$

28. $\frac{2x+3}{2(2x-1)} - \frac{7-x}{2(x+1)} = \frac{7-3x}{4-3x}.$

29. $\frac{12x^3 - 11x^2 + 10x - 78}{8x^2 - 7x + 6} = 1\frac{1}{2}x - \frac{1}{2}.$

30. $\frac{3x-1}{7-x} - \frac{5-4x}{2x+1} = 3.$

250. Literal Quadratics are solved as follows:

(1) Solve the equation $ax^2 + bx = c$.

Multiply the equation by $4a$ and add the square of b,

$$4a^2x^2 + (\ ) + b^2 = 4ac + b^2.$$

Extract the square root, $2ax + b = \pm\sqrt{4ac + b^2}$.

Reduce, $2ax = -b \pm \sqrt{4ac + b^2}$.

$$\therefore\ x = \frac{-b \pm \sqrt{4ac + b^2}}{2a}.$$

(2) Solve the equation $adx - acx^2 = bcx - bd$.

Transpose bcx and change the signs,

$$acx^2 + bcx - adx = bd.$$

Express the left member in *two terms*,

$$acx^2 + (bc - ad)x = bd.$$

Multiply by $4ac$,

$$4a^2c^2x^2 + 4ac(bc - ad)x = 4abcd.$$

Complete the square,

$$4a^2c^2x^2 + (\ ) + (bc - ad)^2 = b^2c^2 + 2abcd + a^2d^2.$$

Extract the root, $2acx + (bc - ad) = \pm(bc + ad)$.

Reduce,
$$2acx = -(bc - ad) \pm (bc + ad)$$
$$= 2ad \text{ or } -2bc.$$

$$\therefore\ x = \frac{d}{c} \text{ or } -\frac{b}{a}.$$

(3) Solve the equation $px^2 - px + qx^2 + qx = \dfrac{pq}{p + q}$.

Express the left member in *two terms*,

$$(p + q)x^2 - (p - q)x = \frac{pq}{p + q}.$$

Multiply by 4 times the coefficient of x^2,

$$4(p + q)^2x^2 - 4(p^2 - q^2)x = 4pq.$$

Complete the square,

$$4(p + q)^2x^2 - (\ ) + (p - q)^2 = p^2 + 2pq + q^2.$$

Extract the root, $2(p + q)x - (p - q) = \pm(p + q)$.

Reduce,
$$2(p + q)x = (p - q) \pm (p + q)$$
$$= 2p \text{ or } -2q.$$

$$\therefore\ x = \frac{p}{p + q} \text{ or } -\frac{q}{p + q}.$$

NOTE. The left-hand member of the equation when simplified must be expressed in *two terms, simple or compound*, one term containing x^2, and the other term containing x.

Exercise 96.

Solve:

1. $x^2 + 2ax = a^2.$

2. $x^2 = 4ax + 7a^2.$

3. $x^2 = \frac{7m^2}{4} - 3mx.$

4. $x^2 - \frac{5nx}{2} - \frac{3n^2}{2} = 0.$

5. $\frac{a^2}{(x+a)^2} = \frac{b^2}{(x-a)^2}.$

6. $cx = ax^2 + bx^2 - \frac{ac}{a+b}.$

7. $\frac{a^2x^2}{b^2} + \frac{b^2}{c^2} = \frac{2ax}{c}.$

8. $(a^2+1)x = ax^2 + a.$

9. $\frac{a}{x-a} + \frac{b}{x-b} = \frac{2c}{x-c}.$

10. $\frac{1}{a+b+x} = \frac{1}{a} + \frac{1}{b} + \frac{1}{x}.$

11. $\frac{1}{a-x} - \frac{1}{a+x} = \frac{3+x^2}{a^2-x^2}.$

12. $\frac{x^2 + 2ab(a^2+b^2)}{a^2+b^2} = 2x.$

13. $\frac{(2x-a)^2}{2x-a+2b} = b.$

14. $x^2 + ax = a + x.$

15. $x^2 + ax = bx + ab.$

16. $\frac{x}{a} + \frac{a}{x} = \frac{x}{b} + \frac{b}{x}.$

17. $\frac{1}{x} + \frac{1}{x+b} = \frac{1}{a} + \frac{1}{a+b}.$

18. $\frac{a}{3} + \frac{5x}{4} - \frac{x^2}{3a} = 0.$

19. $\frac{x+3}{x-3} = a + \frac{x-3}{x+3}.$

20. $mx^2 - 1 = \frac{x(m^3-n^2)}{mn}.$

21. $(ax-b)(bx-a) = c^2.$

22. $\frac{ax+b}{bx+a} = \frac{mx+n}{nx+m}.$

23. $\frac{m}{m+x} + \frac{m}{m-x} = c.$

24. $\frac{(a-1)^2x^2 + 2(3a-1)x}{4a-1} = 1.$

25. $\frac{(a^2-b^2)(x^2+1)}{a^2+b^2} = 2x.$

26. $\frac{x^2 - 4mnx}{(m+n)^2} = (m-n)^2.$

27. $x^2 + \frac{a-b}{ab^2} = \frac{14a^2 - 5ab - 10b^2}{18a^2b^2} + \frac{(2a-3b)x}{2ab}.$

28. $abx^2 + \dfrac{b^2x}{c} = \dfrac{6a^2 + ab - 2b^2}{c^2} - \dfrac{3a^2x}{c}.$

29. $\dfrac{x^2}{3m - 2a} - \dfrac{m^2 - 4a^2}{4a - 6m} = \dfrac{x}{2}.$

30. $6x + \dfrac{(a+b)^2}{x} = 5(a - b) + \dfrac{25ab}{6x}.$

31. $\frac{8}{9}(x^2 + a^2 + ab) = \frac{1}{9}x(20a + 4b).$

32. $x^2 - (b - a)c = ax - bx + cx.$

33. $x^2 - 2mx = (n - p + m)(n - p - m).$

34. $x^2 - (m + n)x = \frac{1}{4}(p + q + m + n)(p + q - m - n).$

35. $mnx^2 - (m + n)(mn + 1)x + (m + n)^2 = 0.$

36. $\dfrac{2b - x - 2a}{bx} + \dfrac{4b - 7a}{ax - bx} = \dfrac{x - 4a}{ab - b^2}.$

37. $2x^2(a^2 - b^2) - (3a^2 + b^2)(x - 1) = (3b^2 + a^2)(x + 1).$

38. $\dfrac{a - 2b - x}{a^2 - 4b^2} - \dfrac{5b - x}{ax + 2bx} + \dfrac{2a - x - 19b}{2bx - ax} = 0.$

39. $\dfrac{x + 13a + 3b}{5a - 3b - x} - 1 = \dfrac{a - 2b}{x + 2b}.$

40. $\dfrac{x + 3b}{8a^2 - 12ab} - \dfrac{3b}{9b^2 - 4a^2} - \dfrac{a + 3b}{(2a + 3b)(x - 3b)} = 0.$

41. $nx^2 + px - px^2 - mx + m - n = 0.$

42. $(a + b + c)x^2 - (2a + b + c)x + a = 0.$

43. $(ax - b)(c - d) = (a - b)(cx - d)x.$

44. $\dfrac{2x + 1}{b} - \dfrac{1}{x}\left(\dfrac{1}{b} - \dfrac{2}{a}\right) = \dfrac{3x + 1}{a}.$

45. $\dfrac{1}{2x^2 + x - 1} + \dfrac{1}{2x^2 - 3x + 1} = \dfrac{a}{2bx - b} - \dfrac{2bx + b}{ax^2 - a}.$

251. Solutions by a Formula. Every affected quadratic may be reduced to the form $x^2+px+q=0$, in which p and q represent numbers, positive or negative, integral or fractional.

Solve: $$x^2+px+q=0.$$
$$4x^2+(\ )+p^2=p^2-4q,$$
$$2x+p=\pm\sqrt{p^2-4q}.$$
$$\therefore\ x=-\frac{p}{2}\pm\frac{1}{2}\sqrt{p^2-4q}.$$

By this formula, the values of x in an equation of the form $x^2+px+q=0$, may be written at once. Thus, take the equation
$$3x^2-5x+2=0.$$

Divide by 3, $$x^2-\frac{5}{3}x+\frac{2}{3}=0.$$

Here, $$p=-\frac{5}{3}, \text{ and } q=\frac{2}{3}.$$
$$\therefore\ x=\frac{5}{6}\pm\frac{1}{2}\sqrt{\frac{25}{9}-\frac{8}{3}},$$
$$=\frac{5}{6}\pm\frac{1}{6}$$
$$=1 \text{ or } \frac{2}{3}.$$

252. Solutions by Factoring. A quadratic which has been reduced to its simplest form, and has all its terms written on one side, may often have that side resolved *by inspection* into factors.

In this case the roots are seen at once without completing the square.

(1) Solve $x^2+7x-60=0.$

Since $x^2+7x-60=(x+12)(x-5),$

the equation $x^2+7x-60=0$

may be written $(x+12)(x-5)=0.$

If *either* of the factors $x + 12$ or $x - 5$ is 0, the *product of the two factors* is 0, and the equation is satisfied.

Hence, $$x + 12 = 0, \text{ or } x - 5 = 0.$$
$$\therefore\ x = -12, \text{ or } x = 5.$$

(2) Solve $x^2 + 7x = 0$.

The equation $$x^2 + 7x = 0$$
becomes $$x(x + 7) = 0,$$
and is satisfied $$\text{if } x = 0, \text{ or if } x + 7 = 0.$$
$\therefore$ the roots are 0 and -7.

This method is easily applied to an equation *all* the terms of which contain x.

(3) Solve $2x^3 - x^2 - 6x = 0$.

The equation $$2x^3 - x^2 - 6x = 0$$
becomes $$x(2x^2 - x - 6) = 0,$$
and is satisfied $$\text{if } x = 0, \text{ or if } 2x^2 - x - 6 = 0.$$

By solving $2x^2 - x - 6 = 0$ the two roots 2 and $-\frac{3}{2}$ are found.

$\therefore$ the equation has *three* roots, 0, 2, $-\frac{3}{2}$.

(4) Solve $x^3 + x^2 - 4x - 4 = 0$.

The equation $$x^3 + x^2 - 4x - 4 = 0$$
becomes $$x^2(x + 1) - 4(x + 1) = 0,$$
$$(x^2 - 4)(x + 1) = 0.$$
$\therefore$ the roots of the equation are -1, 2, -2.

(5) Solve $x^3 - 2x^2 - 11x + 12 = 0$.

We find by trial that if we put 1 for x, the equation is satisfied. Hence, 1 is a root.

Divide by $x - 1$; the given equation may be written
$$(x - 1)(x^2 - x - 12) = 0,$$
and is satisfied $$\text{if } x - 1 = 0, \text{ or if } x^2 - x - 12 = 0.$$

The roots are found to be 1, 4, -3.

(6) Solve the equation $x(x^2 - 9) = a(a^2 - 9)$.

If we put a for x, the equation is satisfied; therefore a is a root (§ 68).

Transpose all the terms to the left member, and divide by $x - a$.

The given equation may be written

$$(x-a)(x^2+ax+a^2-9)=0,$$

and is satisfied if $x-a=0$, or if $x^2+ax+a^2-9=0$.

The roots are found to be

$$a,\ \tfrac{1}{2}(-a+\sqrt{36-3a^2}),\ \tfrac{1}{2}(-a-\sqrt{36-3a^2}).$$

Exercise 97.

Find all the roots of:

1. $(x+1)(x-2)(x^2+x-2)=0.$
2. $(x^2-3x+2)(x^2-x-12)=0.$
3. $(x+1)(x-2)(x+3)=-6.$
4. $2x^3+4x^2-70x=0.$
5. $(x^2-x-6)(x^2-x-20)=0.$
6. $x(x+1)(x+2)=a(a+1)(a+2).$
7. $x^3-x^2-x+1=0.$
8. $8x^3-1=0.$
9. $8x^3+1=0.$
10. $x^6-1=0.$
11. $x(x-a)(x^2-b^2)=0.$
12. $n(x^3+1)+x+1=0.$

253. Equations in the Quadratic Form. An equation is in the *quadratic form* if it contains but two powers of the unknown number, and the exponent of one power is exactly twice that of the other power.

254. Equations not of the second degree, but of the quadratic form, may be solved by completing the square.

(1) Solve: $\qquad 8x^6+63x^3=8.$

We have $\qquad 8x^6+63x^3=8.$

Multiply by 32 and complete the square,

$$256x^6+(\ )+(63)^2=4225.$$

Extract the square root, $\quad 16x^3+63=\pm 65.$

Hence, $\qquad x^3=\dfrac{1}{8}$ or $-8.$

Extracting the cube root, two values of x are $\frac{1}{2}$ and -2. To find the remaining roots, it remains to solve completely the two equations

$$x^3 = \tfrac{1}{8}, \qquad x^3 = -8.$$

We have, $\quad 8x^3 - 1 = 0,$
or, $\quad (2x-1)(4x^2+2x+1) = 0.$
$\therefore$ either $\quad 2x - 1 = 0,$
or, $\quad 4x^2 + 2x + 1 = 0.$

Solving these, we find for three values of x,

$$\tfrac{1}{2},\ \frac{-1+\sqrt{-3}}{4},\ \frac{-1-\sqrt{-3}}{4}.$$

We have, $\quad x^3 + 8 = 0,$
or, $\quad (x+2)(x^2-2x+4) = 0.$
$\therefore$ either $\quad x + 2 = 0,$
or, $\quad x^2 - 2x + 4 = 0.$

Solving these, we find for three values of x,

$$-2,\quad 1+\sqrt{-3},\quad 1-\sqrt{-3}.$$

These six values of x are the six roots of the given equation.

(2) Solve: $$\sqrt{x^3} - 3\sqrt[4]{x^3} = 40.$$

Using fractional exponents, we have $x^{\frac{3}{2}} - 3x^{\frac{3}{4}} = 40.$

Complete the square, $\quad 4x^{\frac{3}{2}} - 12x^{\frac{3}{4}} + 9 = 169.$

Extract the square root, $\quad 2x^{\frac{3}{4}} - 3 = \pm 13.$

$$\therefore\ 2x^{\frac{3}{4}} = 16, \text{ or } -10,$$
$$x^{\frac{3}{4}} = 8, \text{ or } -5,$$
$$x = 16, \text{ or } -5\sqrt[3]{5}.$$

There are other values of x which we do not at present find.

(3) Solve: $$x^2 + \frac{1}{x^2} + x + \frac{1}{x} = 4.$$

Add 2 to both members,

$$x^2 + 2 + \frac{1}{x^2} + x + \frac{1}{x} = 6.$$

Put $x + \frac{1}{x} = y$; the equation becomes

$$y^2 + y = 6.$$

Solving this, $\quad y = 2, \text{ or } -3.$

$$\therefore\ x + \frac{1}{x} = 2, \text{ or } x + \frac{1}{x} = -3.$$

Solving these two equations, we find for the four values of x,

$$1,\ 1,\ \frac{-3+\sqrt{5}}{2},\ \frac{-3-\sqrt{5}}{2}.$$

(4) Solve: $\quad x^4 - 4x^3 + 5x^2 - 2x - 20 = 0.$

Begin by attempting to extract the square root.

$$\begin{array}{r|l}
 & x^4 - 4x^3 + 5x^2 - 2x - 20 \,\underline{|\, x^2 - 2x} \\
 & x^4 \\
2x^2 - 2x & -4x^3 + 5x^2 \\
 & \underline{-4x^3 + 4x^2} \\
 & \qquad\qquad x^2 - 2x - 20.
\end{array}$$

Hence, the equation may be written in the quadratic form

$$(x^2 - 2x)^2 + x^2 - 2x - 20 = 0.$$

Put $x^2 - 2x = y$; the equation becomes

$$y^2 + y - 20 = 0.$$

Solving this, $\qquad y = -5, \text{ or } +4.$

$$\therefore\ x^2 - 2x = -5, \text{ or } x^2 - 2x = 4.$$

Solving these two equations, we find for the four values of x,

$$1 + 2\sqrt{-1},\ 1 - 2\sqrt{-1},\ 1 + \sqrt{5},\ 1 - \sqrt{5}.$$

Exercise 98.

Solve:

1. $x^6 + 7x^3 = 8.$
2. $x^4 - 5x^2 + 4 = 0.$
3. $37x^2 - 9 = 4x^4.$
4. $16x^8 = 17x^4 - 1.$
5. $32x^{10} - 33x^5 + 1 = 0.$
6. $(x^2 - 2)^2 = \frac{1}{4}(x^2 + 12).$
7. $x^{4n} - \dfrac{5x^{2n}}{3} - \dfrac{25}{12} = 0.$
8. $(x^2 - 9)^2 = 3 + 11(x^2 - 2).$
9. $x^6 + 14x^3 + 24 = 0.$
10. $19x^4 + 216x^7 = x.$
11. $x^8 + 22x^4 + 21 = 0.$
12. $x^{2m} + 3x^m - 4 = 0.$
13. $4x^4 - 20x^3 + 23x^2 + 5x = 6.$
14. $\dfrac{1}{x^{2n}} + \dfrac{3}{x^n} - 20 = 0.$
15. $x^4 - 4x^3 - 10x^2 + 28x - 15 = 0.$
16. $x^4 - 2x^3 - 13x^2 + 14x = -24.$
17. $108x^4 = 20x(9x^2 - 1) - 51x^2 + 7.$
18. $(x^2 - 1)(x^2 - 2) + (x^2 - 3)(x^2 - 4) = x^4 + 5.$

255. Radical Equations. If an equation involves radical expressions, we may first clear of radicals as follows:

Solve $\sqrt{x+4}+\sqrt{2x+6}=\sqrt{7x+14}$.

Square both sides,

$$x+4+2\sqrt{(x+4)(2x+6)}+2x+6=7x+14.$$

Transpose and combine, $2\sqrt{(x+4)(2x+6)}=4x+4.$

Divide by 2 and square, $(x+4)(2x+6)=(2x+2)^2.$

Multiply and reduce, $x^2-3x=10.$

Hence, $x=5$, or -2.

Of these two values, only 5 will satisfy the original equation.

The value -2 will satisfy the equation

$$\sqrt{x+4}-\sqrt{2x+6}=\sqrt{7x+14}.$$

In fact, squaring both members of the original equation is equivalent to transposing $\sqrt{7x+14}$ to the left member, and then multiplying by the rationalizing factor $\sqrt{x+4}+\sqrt{2x+6}+\sqrt{7x+14}$, so that the equation stands

$$(\sqrt{x+4}+\sqrt{2x+6}-\sqrt{7x+14})(\sqrt{x+4}+\sqrt{2x+6}+\sqrt{7x+14})=0,$$

which reduces to $\sqrt{(x+4)(2x+6)}-(2x+2)=0.$

Transposing and squaring again is equivalent to multiplying by

$$(\sqrt{x+4}-\sqrt{2x+6}+\sqrt{7x+14})(\sqrt{x+4}-\sqrt{2x+6}-\sqrt{7x+14}).$$

Multiplying and reducing, we have

$$x^2-3x-10=0.$$

Therefore, the equation $x^2-3x-10=0$ is really obtained from

$$\begin{aligned}&(\sqrt{x+4}+\sqrt{2x+6}-\sqrt{7x+14})\\ \times&(\sqrt{x+4}+\sqrt{2x+6}+\sqrt{7x+14})\\ \times&(\sqrt{x+4}-\sqrt{2x+6}-\sqrt{7x+14})\\ \times&(\sqrt{x+4}-\sqrt{2x+6}+\sqrt{7x+14})=0.\end{aligned}$$

This equation is satisfied by any value that will satisfy any one of the *four* factors of its left member. The first factor is satisfied by 5, and the last factor by -2, while no values can be found to satisfy the second or third factor.

Hence, if a radical equation of this form is proposed for solution, if there is a value of x that will satisfy the particular equation given, that value must be retained, and any value that does not satisfy the equation given must be rejected. (See Wentworth, McLellan, and Glashan's Algebraic Analysis, pp. 278–281.)

256. Some radical equations may be solved as follows:

Solve $7x^2 - 5x + 8\sqrt{7x^2 - 5x + 1} = -8.$

Add 1 to both sides,

$$7x^2 - 5x + 1 + 8\sqrt{7x^2 - 5x + 1} = -7.$$

Put $\sqrt{7x^2 - 5x + 1} = y$; the equation becomes

$$y^2 + 8y = -7.$$

Hence,

$$y = -1, \text{ or } -7,$$
$$y^2 = 1, \text{ or } 49.$$

We now have $7x^2 - 5x + 1 = 1$, or $7x^2 - 5x + 1 = 49$.

Solving these, we find for the values of x,

$$0,\ \frac{5}{7} \ \Big|\ 3,\ -\frac{16}{7}.$$

These values all satisfy the given equation when we take the *negative* value of the square root of the expression $7x^2 - 5x + 1$; they are in fact the four roots of the biquadratic obtained by clearing the given equation of radicals.

Exercise 99.

Solve:

1. $x^2 - 3x - 6\sqrt{x^2 - 3x - 3} + 2 = 0.$

2. $x^2 + 3x - \frac{3}{x} + \frac{1}{x^2} = \frac{7}{36}.$

3. $(2x^2 - 3x)^2 - 2(2x^2 - 3x) = 15.$

4. $(ax - b)^2 + 4a(ax - b) = \frac{9a^2}{4}.$

5. $3(2x^2 - x) - (2x^2 - x)^{\frac{1}{2}} = 2.$

6. $15x - 3x^2 + 4(x^2 - 5x + 5)^{\frac{1}{2}} = 16.$

7. $x^2 + x^{-2} + x + x^{-1} = 4.$

8. $x^2 + \sqrt{x^2 - 7} = 19.$

9. $x^2 + x + \frac{1}{6}(x^2 + x)^{\frac{1}{2}} = \frac{7}{6}.$

10. $(x + 1)^{\frac{1}{2}} + (x - 1)^{\frac{1}{2}} = 5.$

11. $x - 1 = 2 + 2x^{-\frac{1}{2}}$.

12. $\sqrt{3x+5} - \sqrt{3x-5} = 4$.

13. $(x^4 + 1) - x(x^2 + 1) = -2x^2$.

14. $2x^2 - 2\sqrt{2x^2 - 5x} = 5(x + 3)$.

15. $x + 2 - 4x\sqrt{x+2} = 12x^2$.

16. $\sqrt{2x+a} + \sqrt{2x-a} = b$.

17. $\sqrt{9x^2 + 21x + 1} - \sqrt{9x^2 + 6x + 1} = 3x$.

18. $x^{\frac{4}{3}} - 4x^{\frac{2}{3}} + x^{-\frac{4}{3}} + 4x^{-\frac{2}{3}} = -\frac{7}{4}$.

19. $\sqrt{x+1} + \sqrt{x+16} = \sqrt{x+25}$.

20. $\sqrt{2x+1} - \sqrt{x+4} = \frac{1}{3}\sqrt{x-3}$.

21. $\sqrt{x+3} + \sqrt{x+8} = 5\sqrt{x}$.

22. $\sqrt{3+x} + \sqrt{x} = \dfrac{6}{\sqrt{3+x}}$.

23. $\sqrt{x^2-1} + 6 = \dfrac{16}{\sqrt{x^2-1}}$.

24. $\dfrac{1}{\sqrt{x+1}} + \dfrac{1}{\sqrt{x-1}} = \dfrac{2}{\sqrt{x^2-1}}$.

25. $\dfrac{\sqrt{x+2a} - \sqrt{x-2a}}{\sqrt{x-2a} + \sqrt{x+2a}} = \dfrac{x}{2a}$.

26. $\dfrac{3x + \sqrt{4x - x^2}}{3x - \sqrt{4x - x^2}} = 2$.

27. $\dfrac{\sqrt{7x^2+4} + 2\sqrt{3x-1}}{\sqrt{7x^2+4} - 2\sqrt{3x-1}} = 7$.

28. $\sqrt{(x-a)^2 + 2ab + b^2} = x - a + b$.

29. $\sqrt{(x+a)^2 + 2ab + b^2} = b - a - x$.

30. $\sqrt{\dfrac{x}{4} + 3} + \sqrt{\dfrac{x}{4} - 3} = \sqrt{\dfrac{2x}{3}}$.

31. $4x^{\frac{1}{2}} - 3(x^{\frac{1}{2}} + 1)(x^{\frac{1}{2}} - 2) = x^{\frac{1}{2}}(10 - 3x^{\frac{1}{2}})$.

32. $(x^{\frac{2}{3}} - 2)(x^{\frac{4}{3}} - 4) = x^{\frac{2}{3}}(x^{\frac{2}{3}} - 1)^2 - 12$.

257. Problems involving Quadratics. Problems which involve quadratic equations apparently have two solutions, since a quadratic equation has two roots. When both roots of the quadratic equation are positive integers, they will give two actual solutions of the problem.

Fractional and negative roots will in some problems give admissible solutions; in other problems they will not give admissible solutions.

No difficulty will be found in selecting the result which belongs to the particular problem we are solving. Sometimes, by a change in the statement of the problem, we may form a new problem which corresponds to the result that was inapplicable to the original problem.

Imaginary roots indicate that the problem is impossible.

Here as in simple equations x stands for an unknown *number*.

(1) The sum of the squares of two consecutive numbers is 481. Find the numbers.

Let $x =$ one number,

and $x + 1 =$ the other.

Then $x^2 + (x+1)^2 = 481,$

or $2x^2 + 2x + 1 = 481.$

The solution of which gives $x = 15$ or -16.

The positive 15 gives for the numbers, 15 and 16.

The negative root -16 is inapplicable to the problem, as *consecutive numbers* are understood to be integers which follow one another in the common scale 1, 2, 3, 4 .....

(2) A pedler bought a number of knives for $2.40. Had he bought 4 more for the same money, he would have paid 3 cents less for each. How many knives did he buy, and what did he pay for each?

Let $x =$ number of knives he bought.

Then $\dfrac{240}{x} =$ number of cents he paid for each.

But if $x + 4 =$ number of knives he bought,

$$\frac{240}{x+4} = \text{number of cents he paid for each,}$$

$$\frac{240}{x} - \frac{240}{x+4} = \text{the difference in price.}$$

But $3 =$ the difference in price.

$$\therefore \frac{240}{x} - \frac{240}{x+4} = 3.$$

Solving, $x = 16$ or -20.

He bought 16 knives, therefore, and paid $\frac{240}{16}$, or 15 cents for each.

If the problem is changed so as to read: A pedler bought a number of knives for \$2.40, and if he had bought 4 *less* for the same money, he would have paid 3 cents *more* for each, the equation will be

$$\frac{240}{x-4} - \frac{240}{x} = 3.$$

Solving, $x = 20$ or -16.

This second problem is therefore the one which the negative answer of the first problem suggests.

(3) What is the price of eggs per dozen when 2 more in a shilling's worth lowers the price 1 penny per dozen?

Let $x =$ number of eggs for a shilling.

Then $\frac{1}{x} =$ cost of one egg in shillings.

and $\frac{12}{x} =$ cost of 1 dozen in shillings.

But if $x + 2 =$ number of eggs for a shilling,

$$\frac{12}{x+2} = \text{cost of 1 dozen in shillings.}$$

$$\therefore \frac{12}{x} - \frac{12}{x+2} = \frac{1}{12} \text{(1 penny being } \tfrac{1}{12} \text{ of a shilling).}$$

The solution of which gives $x = 16$, or -18.

And if 16 eggs cost a shilling, 1 dozen will cost 9 pence.

Therefore, the price of the eggs is 9 pence per dozen.

Exercise 100.

1. The sum of the squares of three consecutive numbers is 365. Find the numbers.

2. Three times the product of two consecutive numbers exceeds four times their sum by 8. Find the numbers.

3. The product of three consecutive numbers is equal to three times the middle number. Find the numbers.

4. A boy bought a number of apples for 16 cents. Had he bought 4 more for the same money he would have paid $\frac{1}{3}$ of a cent less for each apple. How many did he buy?

5. For building 108 rods of stone-wall, 6 days less would have been required if 3 rods more a day had been built. How many rods a day were built?

6. A merchant bought some pieces of silk for $900. Had he bought 3 pieces more for the same money he would have paid $15 less for each piece. How many did he buy?

7. A merchant bought some pieces of cloth for $168.75. He sold the cloth for $12 a piece and gained as much as 1 piece cost him. How much did he pay for each piece?

8. Find the price of eggs per score when 10 more in $62\frac{1}{2}$ cents' worth lowers the price $31\frac{1}{4}$ cents per hundred.

9. The area of a square may be doubled by increasing its length by 6 inches and its breadth by 4 inches. Determine its side.

10. The length of a rectangular field exceeds the breadth by 1 yard, and the area is 3 acres. Determine its dimensions.

11. There are three lines of which two are each $\frac{4}{7}$ of the third, and the sum of the squares described on them is equal to a square yard. Determine the lengths of the lines in inches.

12. A grass plot 9 yards long and 6 yards broad has a path round it. The area of the path is equal to that of the plot. Determine the width of the path.

13. Find the radius of a circle the area of which would be doubled by increasing its radius by 1 inch.

14. Divide a line 20 inches long into two parts so that the rectangle contained by the whole and one part may be equal to the square on the other part.

15. A can do some work in 9 hours less time than B can do it, and together they can do it in 20 hours. How long will it take each alone to do it?

16. A vessel which has two pipes can be filled in 2 hours less time by one than by the other, and by both together in 2 hours 55 minutes. How long will it take each pipe alone to fill the vessel?

17. A vessel which has two pipes can be filled in 2 hours less time by one than by the other, and by both together in 1 hour 52 minutes 30 seconds. How long will it take each pipe alone to fill the vessel?

18. An iron bar weighs 36 pounds. If it had been 1 foot longer each foot would have weighed $\frac{1}{2}$ a pound less. Find the length and the weight per foot.

19. A number is expressed by two digits, the units' digit being the square of the other, and when 54 is added its digits are interchanged. Find the number.

20. Divide 35 into two parts so that the sum of the two fractions formed by dividing each part by the other may be $2\frac{1}{12}$.

21. A boat's crew row $3\frac{1}{2}$ miles down a river and back again in 1 hour 40 minutes. If the current of the river is 2 miles per hour, determine their rate of rowing in still water.

22. A detachment from an army was marching in regular column with 5 men more in depth than in front. On approaching the enemy the front was increased by 845 men, and the whole was thus drawn up in 5 lines. Find the number of men.

23. A jockey sold a horse for \$144, and gained as much per cent as the horse cost. What did the horse cost?

24. A merchant expended a certain sum of money in goods, which he sold again for \$24, and lost as much per cent as the goods cost him. How much did he pay for the goods?

25. A broker bought a number of bank shares (\$100 each), when they were at a certain per cent *discount*, for \$7500; and afterwards when they were at the same per cent *premium*, sold all but 60 for \$5000. How many shares did he buy, and at what price?

26. The thickness of a rectangular solid is $\frac{2}{3}$ of its width, and its length is equal to the sum of its width and thickness; also, the number of cubic yards in its volume added to the number of linear yards in its edges is $\frac{5}{3}$ of the number of square yards in its surface. Determine its dimensions.

27. If a carriage-wheel $16\frac{1}{2}$ feet round took 1 second more to revolve, the rate of the carriage per hour would be $1\frac{7}{8}$ miles less. At what rate is the carriage travelling?

CHAPTER XX.

SIMULTANEOUS QUADRATIC EQUATIONS.

258. Quadratic equations involving two unknown numbers require different methods for their solution, according to the form of the equations.

CASE I.

259. When from One of the Equations the Value of One of the Unknown Numbers can be found in Terms of the Other, and this Value substituted in the Other Equation.

Solve:

$$\left.\begin{aligned} 3x^2 - 2xy &= 5 \\ x - y &= 2 \end{aligned}\right\} \quad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

Transpose x in (2), $\qquad y = x - 2.$

In (1) put $x - 2$ for y.

$$3x^2 - 2x(x - 2) = 5.$$

The solution of which gives $x = 1$, or $x = -5$.

If $\qquad x = 1,$

$\qquad y = 1 - 2 = -1;$

and if $\qquad x = -5,$

$\qquad y = -5 - 2 = -7.$

We have therefore the *pairs* of values,

$$\left.\begin{aligned} x &= 1 \\ y &= -1 \end{aligned}\right\}; \text{ or } \left.\begin{aligned} x &= -5 \\ y &= -7 \end{aligned}\right\}.$$

The original equations are both satisfied by either pair of values. But the values $x = 1$, $y = -7$, will not satisfy the equations; nor will the values $x = -5$, $y = -1$.

The student must be careful to join to each value of x *the corresponding value of y*.

Case II.

260. When the Left Side of Each of the Two Equations is Homogeneous and of the Second Degree.

Solve: $$\left.\begin{aligned} 2y^2 - 4xy + 3x^2 &= 17 \\ y^2 - x^2 &= 16 \end{aligned}\right\} \quad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

Let $y = vx$, and substitute vx for y in both equations.

From (1), $$2v^2x^2 - 4vx^2 + 3x^2 = 17.$$

$$\therefore\ x^2 = \frac{17}{2v^2 - 4v + 3}.$$

From (2), $$v^2x^2 - x^2 = 16.$$

$$\therefore\ x^2 = \frac{16}{v^2 - 1}.$$

Equate the values of x^2, $\dfrac{17}{2v^2 - 4v + 3} = \dfrac{16}{v^2 - 1}$,

$$32v^2 - 64v + 48 = 17v^2 - 17,$$

$$15v^2 - 64v = -65,$$

$$225v^2 - 960v = -975,$$

$$225v^2 - (\ ) + (32)^2 = 49,$$

$$15v - 32 = \pm 7.$$

$$\therefore\ v = \frac{5}{3}, \text{ or } \frac{13}{5}.$$

If $$v = \frac{5}{3},$$

$$y = vx = \frac{5x}{3}.$$

Substitute in (2),

$$\frac{25x^2}{9} - x^2 = 16,$$

$$x^2 = 9,$$

$$x = \pm 3,$$

$$y = \frac{5x}{3} = \pm 5.$$

If $$v = \frac{13}{5},$$

$$y = vx = \frac{13x}{5}.$$

Substitute in (2),

$$\frac{169x^2}{25} - x^2 = 16,$$

$$x^2 = \frac{25}{9},$$

$$x = \pm\frac{5}{3},$$

$$y = \frac{13x}{5} = \pm\frac{13}{3}.$$

Case III.

261. When the Two Equations are Symmetrical with Respect to x and y; that is, when x and y are similarly involved.

Thus, the expressions

$$2x^3+3x^2y^2+2y^3,\ 2xy-3x-3y+1,\ x^4-3x^2y-3xy^2+y^4,$$

are symmetrical expressions. In this case the general rule is to combine the equations in such a manner as to remove the highest powers of x and y.

Solve:
$$\left.\begin{aligned} x^4+y^4&=337 \\ x+y&=7 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

To remove x^4 and y^4, raise (2) to the fourth power.

$$\begin{aligned} &x^4+4x^3y+6x^2y^2+4xy^3+\ \ y^4=2401 \\ \text{Add (1),}\quad &x^4 \qquad\qquad\qquad\qquad\qquad\quad +\ \ y^4=\ \ 337 \\ \hline &2x^4+4x^3y+6x^2y^2+4xy^3+2y^4=2738 \end{aligned}$$

Divide by 2, $\quad x^4+2x^3y+3x^2y^2+2xy^3+y^4=1369.$

Extract the square root, $\quad x^2+xy+y^2=\pm 37. \qquad (3)$

Subtract (3) from $(2)^2$, $\quad xy=12$, or 86.

We now have to solve the two pairs of equations,

$$\left.\begin{aligned} x+y&=7 \\ xy&=12 \end{aligned}\right\};\quad \left.\begin{aligned} x+y&=7 \\ xy&=86 \end{aligned}\right\}.$$

From the first,
$$\left.\begin{aligned} x&=4 \\ y&=3 \end{aligned}\right\};\ \text{or}\ \left.\begin{aligned} x&=3 \\ y&=4 \end{aligned}\right\}.$$

From the second,
$$\left.\begin{aligned} x&=\frac{7\pm\sqrt{-295}}{2} \\ y&=\frac{7\mp\sqrt{-295}}{2} \end{aligned}\right\}.$$

262. The preceding cases are *general methods* for the solution of equations which belong to the kinds referred to; often, however, in the solution of these and other kinds of simultaneous equations involving quadratics, a little ingenuity will suggest some step by which the roots may be found more easily than by the general method.

(1) Solve:

$$\left.\begin{aligned} x+y &= 40 \\ xy &= 300 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Square (1), $x^2+2xy+y^2=1600.$ (3)

Multiply (2) by 4, $4xy=1200.$ (4)

Subtract (4) from (3),

$$x^2-2xy+y^2=400. \qquad (5)$$

Extract root of each side, $x-y=\pm 20.$ (6)

From (1) and (6),

$$\left.\begin{aligned} x&=30 \\ y&=10 \end{aligned}\right\};\ \text{or}\ \left.\begin{aligned} x&=10 \\ y&=30 \end{aligned}\right\}.$$

(2) Solve:

$$\left.\begin{aligned} \frac{1}{x}+\frac{1}{y} &= \frac{9}{20} \\ \frac{1}{x^2}+\frac{1}{y^2} &= \frac{41}{400} \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Square (1), $\frac{1}{x^2}+\frac{2}{xy}+\frac{1}{y^2}=\frac{81}{400}.$ (3)

Subtract (2) from (3), $\frac{2}{xy}=\frac{40}{400}.$ (4)

Subtract (4) from (2),

$$\frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}=\frac{1}{400}.$$

Extract the root, $\frac{1}{x}-\frac{1}{y}=\pm\frac{1}{20}.$ (5)

From (1) and (5),

$$\left.\begin{aligned} x&=4 \\ y&=5 \end{aligned}\right\};\ \text{or}\ \left.\begin{aligned} x&=5 \\ y&=4 \end{aligned}\right\}.$$

(3) Solve:
$$\left.\begin{aligned} x - y &= 4 \\ x^2 + y^2 &= 40 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Square (1), $x^2 - 2xy + y^2 = 16.$ (3)

Subtract (2) from (3), $-2xy = -24.$ (4)

Subtract (4) from (2),

$$x^2 + 2xy + y^2 = 64.$$

Extract the root, $x + y = \pm 8.$ (5)

From (1) and (5),
$$\left.\begin{aligned} x &= 6 \\ y &= 2 \end{aligned}\right\}; \text{ or } \left.\begin{aligned} x &= -2 \\ y &= -6 \end{aligned}\right\}.$$

(4) Solve:
$$\left.\begin{aligned} x^3 + y^3 &= 91 \\ x + y &= 7 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Divide (1) by (2), $x^2 - xy + y^2 = 13.$ (3)

Square (2), $x^2 + 2xy + y^2 = 49.$ (4)

Subtract (3) from (4), $3xy = 36.$

Divide by -3, $-xy = -12.$ (5)

Add (5) and (3), $x^2 - 2xy + y^2 = 1.$

Extract the root, $x - y = \pm 1.$ (6)

From (2) and (6),
$$\left.\begin{aligned} x &= 4 \\ y &= 3 \end{aligned}\right\}; \text{ or } \left.\begin{aligned} x &= 3 \\ y &= 4 \end{aligned}\right\}.$$

(5) Solve:
$$\left.\begin{aligned} x^3 + y^3 &= 18xy \\ x + y &= 12 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Divide (1) by (2), $x^2 - xy + y^2 = \frac{3xy}{2}.$ (3)

Square (2), $x^2 + 2xy + y^2 = 144.$ (4)

Subtract (4) from (3), $-3xy = \frac{3xy}{2} - 144,$

which gives $xy = 32.$

We now have,
$$\left.\begin{aligned} x + y &= 12 \\ xy &= 32 \end{aligned}\right\}.$$

Solving, we find,
$$\left.\begin{aligned} x &= 8 \\ y &= 4 \end{aligned}\right\}; \text{ or } \left.\begin{aligned} x &= 4 \\ y &= 8 \end{aligned}\right\}.$$

Exercise 101.

Solve:

1. $\left.\begin{array}{l} x+y=13 \\ xy=36 \end{array}\right\}$

2. $\left.\begin{array}{l} x+y=29 \\ xy=100 \end{array}\right\}$

3. $\left.\begin{array}{l} x-y=19 \\ xy=66 \end{array}\right\}$

4. $\left.\begin{array}{l} x-y=45 \\ xy=250 \end{array}\right\}$

5. $\left.\begin{array}{l} x-y=10 \\ x^2+y^2=178 \end{array}\right\}$

6. $\left.\begin{array}{l} x-y=14 \\ x^2+y^2=436 \end{array}\right\}$

7. $\left.\begin{array}{l} x+y=12 \\ x^2+y^2=104 \end{array}\right\}$

8. $\left.\begin{array}{l} \frac{1}{x}+\frac{1}{y}=\frac{3}{4} \\ \frac{1}{x^2}+\frac{1}{y^2}=\frac{5}{16} \end{array}\right\}$

9. $\left.\begin{array}{l} \frac{1}{x}+\frac{1}{y}=5 \\ \frac{1}{x^2}+\frac{1}{y}=13 \end{array}\right\}$

10. $\left.\begin{array}{l} 7x^2-8xy=159 \\ 5x+2y=7 \end{array}\right\}$

11. $\left.\begin{array}{l} x+y=49 \\ x^2+y^2=1681 \end{array}\right\}$

12. $\left.\begin{array}{l} x^3+y^3=341 \\ x+y=11 \end{array}\right\}$

13. $\left.\begin{array}{l} x^3+y^3=1008 \\ x+y=12 \end{array}\right\}$

14. $\left.\begin{array}{l} x^3-y^3=98 \\ x-y=2 \end{array}\right\}$

15. $\left.\begin{array}{l} x^3-y^3=279 \\ x-y=3 \end{array}\right\}$

16. $\left.\begin{array}{l} x-3y=1 \\ xy+y^2=5 \end{array}\right\}$

17. $\left.\begin{array}{l} 4y=5x+1 \\ 2xy=33-x^2 \end{array}\right\}$

18. $\left.\begin{array}{l} \frac{1}{x}-\frac{1}{y}=3 \\ \frac{1}{x^2}-\frac{1}{y^2}=21 \end{array}\right\}$

19. $\left.\begin{array}{l} \frac{1}{x}-\frac{1}{y}=2\frac{1}{2} \\ \frac{1}{x^2}-\frac{1}{y^2}=8\frac{3}{4} \end{array}\right\}$

20. $\left.\begin{array}{l} x^2-2xy-y^2=1 \\ x+y=2 \end{array}\right\}$

Exercise 102.

Solve:

1. $\left.\begin{array}{l} x^2 + xy + 2y^2 = 74 \\ 2x^2 + 2xy + y^2 = 73 \end{array}\right\}$

2. $\left.\begin{array}{l} x^2 + xy + 4y^2 = 6 \\ 3x^2 + 8y^2 = 14 \end{array}\right\}$

3. $\left.\begin{array}{l} x^2 - xy + y^2 = 21 \\ y^2 - 2xy = -15 \end{array}\right\}$

4. $\left.\begin{array}{l} x^2 - 4y^2 - 9 = 0 \\ xy + 2y^2 - 3 = 0 \end{array}\right\}$

5. $\left.\begin{array}{l} x^2 - xy - 35 = 0 \\ xy + y^2 - 18 = 0 \end{array}\right\}$

6. $\left.\begin{array}{l} x^2 + xy + 2y^2 = 44 \\ 2x^2 - xy + y^2 = 16 \end{array}\right\}$

7. $\left.\begin{array}{l} x^2 + xy - 15 = 0 \\ xy - y^2 - 2 = 0 \end{array}\right\}$

8. $\left.\begin{array}{l} x^2 - xy + y^2 = 7 \\ 3x^2 + 13xy + 8y^2 = 162 \end{array}\right\}$

9. $\left.\begin{array}{l} 2x^2 + 3xy + y^2 = 70 \\ 6x^2 + xy - y^2 = 50 \end{array}\right\}$

10. $\left.\begin{array}{l} x^2 - xy - y^2 = 5 \\ 2x^2 + 3xy + y^2 = 28 \end{array}\right\}$

11. $\left.\begin{array}{l} 4xy = 96 - x^2y^2 \\ x + y = 6 \end{array}\right\}$

12. $\left.\begin{array}{l} x^2 + y^2 = 18 - x - y \\ xy = 6 \end{array}\right\}$

13. $\left.\begin{array}{l} 2(x^2 + y^2) = 5xy \\ 4(x - y) = xy \end{array}\right\}$

14. $\left.\begin{array}{l} 4(x + y) = 3xy \\ x + y + x^2 + y^2 = 26 \end{array}\right\}$

15. $\left.\begin{array}{l} 4x^2 + xy + 4y^2 = 58 \\ 5x^2 + 5y^2 = 65 \end{array}\right\}$

16. $\left.\begin{array}{l} xy(x + y) = 30 \\ x^3 + y^3 = 35 \end{array}\right\}$

Exercise 103.

Solve:

1. $\left.\begin{array}{l} x - y = 7 \\ x^2 + xy + y^2 = 13 \end{array}\right\}$

2. $\left.\begin{array}{l} x^2 + xy = 35 \\ xy - y^2 = 6 \end{array}\right\}$

3. $\left.\begin{array}{l} xy - 12 = 0 \\ x - 2y = 5 \end{array}\right\}$

4. $\left.\begin{array}{l} xy - 7 = 0 \\ x^2 + y^2 = 50 \end{array}\right\}$

5. $\left.\begin{array}{l} 2x - 5y = 9 \\ x^2 - xy + y^2 = 7 \end{array}\right\}$

6. $\left.\begin{array}{l} x - y = 9 \\ xy + 8 = 0 \end{array}\right\}$

7. $\left.\begin{array}{l} 5x - 7y = 0 \\ 5x^2 - \dfrac{13xy}{4} = 4 - 7y^2 \end{array}\right\}$

8. $\left.\begin{array}{l} x-y=1 \\ x^2+y^2=8\frac{1}{2} \end{array}\right\}$

9. $\left.\begin{array}{l} x^2+4xy=3 \\ 4xy+y^2=2\frac{1}{4} \end{array}\right\}$

10. $\left.\begin{array}{l} x^2-xy+y^2=48 \\ x-y-8=0 \end{array}\right\}$

11. $\left.\begin{array}{l} x^2+3xy+y^2=1 \\ 3x^2+xy+3y^2=13 \end{array}\right\}$

12. $\left.\begin{array}{l} x^2-2xy+3y^2=1\frac{2}{9} \\ x^2+xy-y^2=\frac{1}{9} \end{array}\right\}$

13. $\left.\begin{array}{l} x+y=a \\ 4xy-a^2=-4b^2 \end{array}\right\}$

14. $\left.\begin{array}{l} x-y=1 \\ \dfrac{x}{y}+\dfrac{y}{x}=2\frac{1}{6} \end{array}\right\}$

15. $\left.\begin{array}{l} x^2+9xy=340 \\ 7xy-y^2=171 \end{array}\right\}$

16. $\left.\begin{array}{l} x+y=6 \\ x^3+y^3=72 \end{array}\right\}$

17. $\left.\begin{array}{l} 3xy+2x+y=485 \\ 3x-2y=0 \end{array}\right\}$

18. $\left.\begin{array}{l} x-y=1 \\ x^3-y^3=19 \end{array}\right\}$

19. $\left.\begin{array}{l} x^3+y^3=2728 \\ x^2-xy+y^2=124 \end{array}\right\}$

20. $\left.\begin{array}{l} x+y=a \\ x^2+y^2=b^2 \end{array}\right\}$

21. $\left.\begin{array}{l} x^2-y^2=0 \\ 3x^2-4xy+5y^2=9 \end{array}\right\}$

22. $\left.\begin{array}{l} \dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}=\dfrac{10}{3} \\ x^2+y^2=45 \end{array}\right\}$

23. $\left.\begin{array}{l} \dfrac{1}{x}+\dfrac{1}{y}=5 \\ \dfrac{1}{x+1}+\dfrac{1}{y+1}=\dfrac{17}{12} \end{array}\right\}$

24. $\left.\begin{array}{l} x^2-xy+y^2=7 \\ x^4+x^2y^2+y^4=133 \end{array}\right\}$

25. $\left.\begin{array}{l} x+y=4 \\ x^4+y^4=82 \end{array}\right\}$

26. $\left.\begin{array}{l} x^3-y^3=a^3 \\ x-y=a \end{array}\right\}$

27. $\left.\begin{array}{l} x^2-xy=a^2+b^2 \\ xy-y^2=2ab \end{array}\right\}$

28. $\left.\begin{array}{l} x^2-y^2=4ab \\ xy=a^2-b^2 \end{array}\right\}$

29. $\left.\begin{array}{l} xy=0 \\ x^2+y^2=16 \end{array}\right\}$

30. $\left.\begin{array}{l} x^2+xy+y^2=37 \\ x^4+x^2y^2+y^4=481 \end{array}\right\}$

31. $\left.\begin{array}{l} x^2=ax+by \\ y^2=ay+bx \end{array}\right\}$

32. $\left.\begin{array}{l} x-y-2=0 \\ 15(x^2-y^2)=16xy \end{array}\right\}$

33. $\left.\begin{array}{l} \dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}=\dfrac{89}{40} \\ 6x=20y+9 \end{array}\right\}$

34. $\left.\begin{aligned} \frac{x}{a}+\frac{y}{b}=1 \\ \frac{a}{x}+\frac{b}{y}=4 \end{aligned}\right\}$

35. $\left.\begin{aligned} x^2+y^2=7+xy \\ x^3+y^3=6xy-1 \end{aligned}\right\}$

36. $\left.\begin{aligned} x^5-y^5=3093 \\ x-y=3 \end{aligned}\right\}$

37. $\left.\begin{aligned} \tfrac{3}{5}(x-1)-\tfrac{2}{3}(x+1)(y-1)=-11 \\ \tfrac{1}{3}(y+2)=\tfrac{1}{4}(x+2) \end{aligned}\right\}$

38. $\left.\begin{aligned} 10x^2+15xy=3ab-2a^2 \\ 10y^2+15xy=3ab-2b^2 \end{aligned}\right\}$

39. $\left.\begin{aligned} (2x+3y)^2-2(2x+3y)=8 \\ x^2-y^2=21 \end{aligned}\right\}$

40. $\left.\begin{aligned} x+y+\sqrt{x+y}=a \\ x-y+\sqrt{x-y}=b \end{aligned}\right\}$

41. $\left.\begin{aligned} x^4-x^2y^2+y^4=13 \\ x^2-xy+y^2=3 \end{aligned}\right\}$

42. $\left.\begin{aligned} x^2+y^2+x+y=48 \\ xy=12 \end{aligned}\right\}$

43. $\left.\begin{aligned} x^2+xy+y^2=a^2 \\ x+\sqrt{xy}+y=b \end{aligned}\right\}$

44. $\left.\begin{aligned} (x-y)^2-3(x-y)=10 \\ x^2y^2-3xy=54 \end{aligned}\right\}$

45. $\left.\begin{aligned} \sqrt{x}-\sqrt{y}=x^{\frac{1}{2}}(x^{\frac{1}{2}}+y^{\frac{1}{2}}) \\ (x+y)^2=2(x-y)^2 \end{aligned}\right\}$

46. $\left.\begin{aligned} \left(\frac{3x}{x+y}\right)^{\frac{1}{2}}+\left(\frac{x+y}{3x}\right)^{\frac{1}{2}}=2 \\ xy-(x+y)=54 \end{aligned}\right\}$

47. $\left.\begin{aligned} x+y+\sqrt{xy}=28 \\ x^2+y^2+xy=336 \end{aligned}\right\}$

Exercise 104.

1. If the length and breadth of a rectangle were each increased by 1, the area would be 48; if they were each diminished by 1, the area would be 24. Find length and breadth.

2. The sum of the squares of the two digits of a number is 25, and the product of the digits is 12. Find the number.

3. The sum, the product, and the difference of the squares of two numbers are all equal. Find the numbers.

NOTE. Represent the numbers by $x + y$ and $x - y$, respectively.

4. The difference of two numbers is $\frac{3}{8}$ of the greater, and the sum of their squares is 356. What are the numbers?

5. The numerator and denominator of one fraction are each greater by 1 than those of another, and the sum of the two fractions is $1\frac{5}{12}$; if the numerators were interchanged the sum of the fractions would be $1\frac{1}{2}$. Find the fractions.

6. A man starts from the foot of a mountain to walk to its summit. His rate of walking during the second half of the distance is $\frac{1}{2}$ mile per hour less than his rate during the first half, and he reaches the summit in $5\frac{1}{2}$ hours. He descends in $3\frac{3}{4}$ hours, by walking 1 mile more per hour than during the first half of the ascent. Find the distance to the top and the rates of walking.

NOTE. Let $2x$ = the distance, and y miles per hour = the rate at first. Then $\frac{x}{y} + \frac{x}{y - \frac{1}{2}} = 5\frac{1}{2}$ hours, and $\frac{2x}{y + 1} = 3\frac{3}{4}$ hours.

7. The sum of two numbers which are formed by the same two digits in reverse order is $\frac{55}{18}$ of their difference; and the difference of the squares of the numbers is 3960. Determine the numbers.

8. The hypotenuse of a right triangle is 20, and the area of the triangle is 96. Determine the sides.

NOTE. The square on the hypotenuse = sum of the squares on the legs; and the area of a right triangle = $\frac{1}{2}$ product of legs.

9. Two boys run in opposite directions around a rectangular field, the area of which is an acre; they start from one corner and meet 13 yards from the opposite corner; and the rate of one is $\frac{5}{6}$ of the rate of the other. Determine the dimensions of the field.

10. A, in running a race with B, to a post and back, met him 10 yards from the post. To make it a dead heat, B must have increased his rate from this point $41\frac{3}{7}$ yards per minute; and if, without changing his pace, he had turned back on meeting A, he would have come 4 seconds after him. How far was it to the post?

NOTE. If $2x =$ the number of yards to the post and back, and y the number of yards A runs a minute, then $\frac{x-10}{x+10}$ of y, or $\frac{xy-10y}{x+10} =$ the number of yards B runs a minute.

11. The fore wheel of a carriage turns in a mile 132 times more than the hind wheel; but if the circumferences were each increased by 2 feet, it would turn only 88 times more. Find the circumference of each.

12. A person has \$6500, which he divides into two parts and loans at *different rates* of interest, so that the two parts produce *equal* returns. If the first part had been loaned at the second rate of interest, it would have produced \$180; and if the second part had been loaned at the first rate of interest, it would have produced \$245. Find the rates of interest.

CHAPTER XXI.

PROPERTIES OF QUADRATICS.

263. Every affected quadratic can be reduced to the form $ax^2 + bx + c = 0$, the solution of which gives the two roots

$$-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a} \text{ and } -\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}.$$

CHARACTER OF THE ROOTS.

264. As regards the character of the two roots, there are three cases to be distinguished.

I. If $b^2 - 4ac$ is Positive and not Zero. In this case the roots are *real* and *unequal*. The roots are real, since the square root of a positive number can be found exactly or approximately. If $b^2 - 4ac$ is a perfect square, the roots are rational; if $b^2 - 4ac$ is not a perfect square, the roots are surds.

The roots are unequal, since $\sqrt{b^2 - 4ac}$ is not zero.

II. If $b^2 - 4ac$ is Zero. In this case the two roots are *real* and *equal*, since they both become $-\frac{b}{2a}$.

III. If $b^2 - 4ac$ is Negative. In this case the roots are *imaginary*, since they both involve the square root of a negative number.

The two imaginary roots of a quadratic cannot be equal, since $b^2 - 4ac$ is not zero. They have, however, the same

real part, $-\frac{b}{2a}$, and the same imaginary part, but with opposite signs; such expressions are called **conjugate imaginaries.** The expression $b^2 - 4ac$ is called the **discriminant** of the expression $ax^2 + bx + c$.

265. The above cases may also be distinguished as follows:

CASE I. $b^2 - 4ac > 0$, roots real and unequal.
CASE II. $b^2 - 4ac = 0$, roots real and equal.
CASE III. $b^2 - 4ac < 0$, roots imaginary.

266. By calculating the value of $b^2 - 4ac$, we can determine the character of the roots of a given equation without solving the equation.

(1) $x^2 - 5x + 6 = 0.$

Here $$a = 1,\ b = -5,\ c = 6.$$
$$b^2 - 4ac = 25 - 24 = 1.$$

The roots are real and unequal, and rational.

(2) $3x^2 + 7x - 1 = 0.$

Here $$a = 3,\ b = 7,\ c = -1.$$
$$b^2 - 4ac = 49 + 12 = 61.$$

The roots are real and unequal, and are both surds.

(3) $4x^2 - 12x + 9 = 0.$

Here $$a = 4,\ b = -12,\ c = 9.$$
$$b^2 - 4ac = 144 - 144 = 0.$$

The roots are real and equal.

(4) $2x^2 - 3x + 4 = 0.$

Here $$a = 2,\ b = -3,\ c = 4.$$
$$b^2 - 4ac = 9 - 32 = -23.$$

The roots are both imaginary.

(5) Find the values of m for which the following equation has its two roots equal:

$$2mx^2 + (5m+2)x + (4m+1) = 0.$$

Here $\quad a = 2m,\ b = 5m+2,\ c = 4m+1.$

If the roots are to be equal, we must have

$$b^2 - 4ac = 0, \text{ or } (5m+2)^2 - 8m(4m+1) = 0.$$

This gives $\quad m = 2, \text{ or } -\frac{2}{7}.$

For these values of m the equation becomes

$$4x^2 + 12x + 9 = 0, \text{ and } 4x^2 - 4x + 1 = 0,$$

each of which has its roots equal.

Exercise 105.

Determine without solving the character of the roots of each of the following equations:

1. $x^2 - 7x + 12 = 0.$
2. $x^2 - 7x - 30 = 0.$
3. $x^2 + 4x - 5 = 0.$
4. $5x^2 + 8 = 0.$
5. $7x^2 - 3x - 22 = 0.$
6. $x^2 + 4x + 1 = 0.$
7. $x^2 - 2x + 9 = 0.$
8. $3x^2 - 4x - 4 = 0.$
9. $x^2 + 4x + 4 = 0.$
10. $7x - 3x^2 - 2 = 0.$

Determine the values of m for which the two roots of each of the following equations are equal:

11. $(m+1)x^2 + (m-1)x + m + 1 = 0.$
12. $(3m+1)x^2 + (2m+2)x + m = 0.$
13. $(m-2)x^2 + (m-5)x + 2m - 5 = 0.$
14. $2mx^2 + x^2 - 6mx - 6x + 6m + 1 = 0.$
15. $mx^2 + 2x^2 + 2m = 3mx - 9x + 10.$

Relations of Roots and Coefficients.

267. Consider the equation $x^2 - 10x + 24 = 0$. Resolve into factors, $(x-6)(x-4) = 0$. The two values of x are 6 and 4; their sum is 10, the coefficient of x with its sign changed; their product is 24, the third term.

268. In general, representing the roots of the quadratic equation $ax^2 + bx + c = 0$ by r_1 and r_2, we have (§ 263),

$$r_1 = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a},$$

$$r_2 = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}.$$

Adding, $\quad r_1 + r_2 = -\frac{b}{a};$

multiplying, $\quad r_1 r_2 = \frac{c}{a}.$

If we divide the equation $ax^2 + bx + c = 0$ by a, we have the equation $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$; this may be written $x^2 + px + q = 0$ where $p = \frac{b}{a}$, $q = \frac{c}{a}$.

It appears, then, that if any quadratic equation is made to assume the form $x^2 + px + q = 0$, the following relations hold between the coefficients and roots of the equation:

(1) The **sum** of the two roots is equal to the coefficient of x with its sign changed.

(2) The **product** of the two roots is equal to the constant term.

Thus, the sum of the two roots of the equation

$$x^2 - 7x + 8 = 0$$

is 7, and the product of the roots 8.

269. Resolution into Factors. By § 268, if r_1 and r_2 are the roots of the equation $x^2 + px + q = 0$, the equation may be written

$$x^2 - (r_1 + r_2)x + r_1 r_2 = 0.$$

The left member is the product of $x - r_1$, and $x - r_2$, so that the equation may be also written

$$(x - r_1)(x - r_2) = 0.$$

It appears, then, that the factors of the *quadratic expression* $x^2 + px + q$ are $x - r_1$ and $x - r_2$, where r_1 and r_2 are the roots of the *quadratic equation* $x^2 + px + q = 0$.

The factors are real and different, real and alike, or imaginary, according as r_1 and r_2 are real and unequal, real and equal, or imaginary.

If $r_2 = r_1$, the equation becomes $(x - r_1)(x - r_1) = 0$, or $(x - r_1)^2 = 0$; if, then, the two roots of a quadratic equation are equal, the left member, when all the terms are transposed to that member, will be a perfect square as regards x.

270. If the equation is in the form $ax^2 + bx + c = 0$, the left member may be written

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right),$$

$$a(x - r_1)(x - r_2).$$

271. If the roots of a quadratic equation are given, we can readily form the equation.

Form the equation of which the roots are 3 and $-\frac{5}{2}$.

The equation is $\quad (x - 3)\left(x + \frac{5}{2}\right) = 0,$

or $\quad (x - 3)(2x + 5) = 0,$

or $\quad 2x^2 - x - 15 = 0.$

272. Any quadratic expression may be resolved into factors by putting the expression equal to zero, and solving the equation thus formed.

(1) Resolve into two factors $x^2 - 5x + 3$.

Write the equation

$$x^2 - 5x + 3 = 0.$$

Solve this equation, and the roots are found to be

$$\frac{5+\sqrt{13}}{2} \text{ and } \frac{5-\sqrt{13}}{2}.$$

Therefore, the factors of $x^2 - 5x + 3$ are

$$x - \frac{5+\sqrt{13}}{2} \text{ and } x - \frac{5-\sqrt{13}}{2}.$$

(2) Resolve into factors $3x^2 - 4x + 5$.

Write the equation

$$3x^2 - 4x + 5 = 0.$$

Solve this equation, and the roots are found to be

$$\frac{2+\sqrt{-11}}{3} \text{ and } \frac{2-\sqrt{-11}}{3}.$$

Therefore, the expression $3x^2 - 4x + 5$ may be written (§ 270),

$$3\left(x - \frac{2+\sqrt{-11}}{3}\right)\left(x - \frac{2-\sqrt{-11}}{3}\right).$$

Exercise 106.

Form the equations of which the roots are

1. $2, 1$.
2. $7, -3$.
3. $\frac{1}{2}, \frac{1}{3}$.
4. $\frac{2}{3}, -\frac{3}{2}$.
5. $-5, -\frac{1}{2}$.
6. $-\frac{7}{9}, \frac{9}{7}$.
7. $a - 2b, 3a + 2b$.
8. $2a - b, b - 3a$.
9. $a + 1, 1 - a$.

Resolve into factors:

10. $3x^2 - 15x - 42$.
11. $9x^2 - 27x - 70$.
12. $49x^2 + 49x + 6$.
13. $x^2 - 3x + 4$.
14. $x^2 + x + 1$.
15. $4x^2 + 12x + 13$.

CHAPTER XXII.

RATIO, PROPORTION, AND VARIATION.

273. Ratio of Numbers. The relative magnitude of two numbers is called their **ratio** when expressed by the indicated quotient of the first by the second. Thus, the ratio of a to b is $\frac{a}{b}$, or $a \div b$, or $a : b$.

The first term of a ratio is called the **antecedent**, and the second term the **consequent.** When the antecedent is equal to the consequent, the ratio is called a ratio of **equality**; when the antecedent is greater than the consequent, the ratio is called a ratio of **greater inequality**; when less, a ratio of **less inequality.**

When the antecedent and consequent are interchanged, the resulting ratio is called the **inverse** of the given ratio. Thus, the ratio 3 : 6 is the *inverse* of the ratio 6 : 3.

274. A ratio will not be altered if both its terms are multiplied by the same number. For the ratio $a : b$ is represented by $\frac{a}{b}$, the ratio $ma : mb$ is represented by $\frac{ma}{mb}$; and since $\frac{ma}{mb} = \frac{a}{b}$, we have $ma : mb = a : b$.

275. A ratio will be altered if different multipliers of its terms are taken; and will be increased or diminished according as the multiplier of the antecedent is greater than or less than that of the consequent. Thus,

If	$m > n,$	If	$m < n,$
then	$ma > na,$	then	$ma < na,$
and	$\dfrac{ma}{nb} > \dfrac{na}{nb};$	and	$\dfrac{ma}{nb} < \dfrac{na}{nb};$
but	$\dfrac{na}{nb} = \dfrac{a}{b}.$	but	$\dfrac{na}{nb} = \dfrac{a}{b}.$
	$\therefore \dfrac{ma}{nb} > \dfrac{a}{b},$		$\therefore \dfrac{ma}{nb} < \dfrac{a}{b},$
or	$ma : nb > a : b.$	or	$ma : nb < a : b.$

276. Ratios are *compounded* by taking the product of the fractions that represent them. Thus, the ratio compounded of $a : b$ and $c : d$ is $ac : bd$.

The ratio compounded of $a : b$ and $a : b$ is the *duplicate* ratio $a^2 : b^2$; the ratio compounded of $a : b$, $a : b$, and $a : b$ is the *triplicate* ratio $a^3 : b^3$; and so on.

277. Ratios are *compared* by comparing the fractions that represent them.

Thus, $\qquad a : b > \text{or} < c : d,$

according as $\qquad \dfrac{a}{b} > \text{or} < \dfrac{c}{d},$

as $\qquad \dfrac{ad}{bd} > \text{or} < \dfrac{bc}{bd},$

as $\qquad ad > \text{or} < bc.$

278. Proportion of Numbers. Four numbers, a, b, c, d, are said to be in **proportion** when the ratio $a : b$ is equal to the ratio $c : d$.

We then write $a : b = c : d$, and read this, the ratio of a to b equals the ratio of c to d, or a is to b as c is to d.

A proportion is also written $a : b :: c : d$.

The four numbers, a, b, c, d, are called **proportionals**; a and d are called the **extremes**, b and c the **means**.

279. When four numbers are in proportion, the product of the extremes is equal to product of the means.

For, if $$a : b = c : d,$$

then $$\frac{a}{b} = \frac{c}{d}.$$

Multiplying by bd, $$ad = bc.$$

The equation $ad = bc$ gives $a = \frac{bc}{d}$, $b = \frac{ad}{c}$; so that an extreme may be found by dividing the product of the means by the other extreme; and a mean may be found by dividing the product of the extremes by the other mean. If three terms of a proportion are given, it appears from the above that the fourth term can have one, and but one, value.

280. If the product of two numbers is equal to the product of two others, either two may be made the extremes of a proportion and the other two the means.

For, if $$ad = bc,$$

then, dividing by bd, $$\frac{ad}{bd} = \frac{bc}{bd},$$

or $$\frac{a}{b} = \frac{c}{d}.$$

$$\therefore\ a : b = c : d.$$

281. Transformations of a Proportion. If four numbers, a, b, c, d, are in proportion, they will be in proportion by:

I. Inversion; that is, b will be to a as d is to c.

For, if $$a : b = c : d,$$

then $$\frac{a}{b} = \frac{c}{d},$$

and $$1 \div \frac{a}{b} = 1 \div \frac{c}{d},$$

or $$\frac{b}{a} = \frac{d}{c}.$$

$$\therefore\ b : a = d : c.$$

II. Composition; that is, $a + b$ will be to b as $c + d$ is to d.

For, if $$a : b = c : d,$$

then $$\frac{a}{b} = \frac{c}{d},$$

and $$\frac{a}{b} + 1 = \frac{c}{d} + 1,$$

or $$\frac{a+b}{b} = \frac{c+d}{d}.$$

$$\therefore\ a + b : b = c + d : d.$$

III. Division; that is, $a - b$ will be to b as $c - d$ is to d.

For, if $$a : b = c : d,$$

then $$\frac{a}{b} = \frac{c}{d},$$

and $$\frac{a}{b} - 1 = \frac{c}{d} - 1,$$

or $$\frac{a-b}{b} = \frac{c-d}{d}.$$

$$\therefore\ a - b : b = c - d : d.$$

IV. Composition and Division; that is, $a + b$ will be to $a - b$ as $c + d$ is to $c - d$.

For, from II., $$\frac{a+b}{b} = \frac{c+d}{d},$$

and from III., $$\frac{a-b}{b} = \frac{c-d}{d}.$$

Dividing, $$\frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

$$\therefore\ a + b : a - b = c + d : c - d.$$

V. Alternation; that is, a will be to c as b is to d.

For, if $$a : b = c : d,$$

then $$\frac{a}{b} = \frac{c}{d}.$$

Multiplying by $\frac{b}{c}$, $$\frac{ab}{bc} = \frac{bc}{cd},$$

or $$\frac{a}{c} = \frac{b}{d}.$$

$$\therefore\ a : c = b : d.$$

282. In a Series of Equal Ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

For, if $$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h},$$

r may be put for each of these ratios.

Then $$\frac{a}{b} = r,\ \frac{c}{d} = r,\ \frac{e}{f} = r,\ \frac{g}{h} = r.$$

$$\therefore\ a = br,\ c = dr,\ e = fr,\ g = hr.$$

$$\therefore\ a + c + e + g = (b + d + f + h)r.$$

$$\therefore\ \frac{a + c + e + g}{b + d + f + h} = r = \frac{a}{b}.$$

$$\therefore\ a + c + e + g : b + d + f + h = a : b.$$

In like manner it may be shown that

$$ma + nc + pe + qg : mb + nd + pf + qh = a : b.$$

283. Continued Proportion. Numbers are said to be in *continued proportion* when the first is to the second as the second is to the third, and so on. Thus, a, b, c, d, are in continued proportion when $\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$.

284. If a, b, c are proportionals, so that $a : b = b : c$, then b is called a **mean proportional** between a and c, and c is called a **third proportional** to a and b.

If $a : b = b : c$, then $b = \sqrt{ac}$.

For, if
$$a : b = b : c,$$
then
$$\frac{a}{b} = \frac{b}{c},$$
and
$$b^2 = ac.$$
$$\therefore\ b = \sqrt{ac}.$$

285. The products of the corresponding terms of two or more proportions are in proportion.

For, if
$$a : b = c : d,$$
$$e : f = g : h,$$
$$k : l = m : n,$$
then
$$\frac{a}{b} = \frac{c}{d},\quad \frac{e}{f} = \frac{g}{h},\quad \frac{k}{l} = \frac{m}{n}.$$

Taking the product of the left members, and also of the right members of these equations,
$$\frac{aek}{bfl} = \frac{cgm}{dhn}.$$
$$\therefore\ aek : bfl = cgm : dhn.$$

286. Like powers, or like roots, of the terms of a proportion are in proportion.

For, if
$$a : b = c : d,$$
then
$$\frac{a}{b} = \frac{c}{d}.$$

Raising both sides to the nth power,
$$\frac{a^n}{b^n} = \frac{c^n}{d^n}.$$
$$\therefore\ a^n : b^n = c^n : d^n.$$

Extracting the nth root, $\dfrac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \dfrac{c^{\frac{1}{n}}}{d^{\frac{1}{n}}}$.

$$\therefore\ a^{\frac{1}{n}} : b^{\frac{1}{n}} = c^{\frac{1}{n}} : d^{\frac{1}{n}}.$$

287. The laws that have been established for ratios should be remembered when ratios are expressed in fractional form.

(1) Solve: $\dfrac{x^2+x+1}{x^2-x-1} = \dfrac{x^2-x+2}{x^2+x-2}$.

By composition and division,

$$\frac{2x^2}{2(x+1)} = \frac{2x^2}{-2(x-2)},$$

This equation is satisfied when $x = 0$. For any other value of x, we may divide by x^2.

We then have $$\frac{1}{x+1} = \frac{1}{2-x},$$

and therefore $$x = \tfrac{1}{2}.$$

(2) If $a : b = c : d$, show that

$$a^2 + ab : b^2 - ab = c^2 + cd : d^2 - cd.$$

If $$\frac{a}{b} = \frac{c}{d},$$

then $$\frac{a+b}{a-b} = \frac{c+d}{c-d},$$

and $$\frac{a}{-b} = \frac{c}{-d}.$$

$$\therefore\ \frac{a}{-b} \times \frac{a+b}{a-b} = \frac{c}{-d} \times \frac{c+d}{c-d};$$

that is, $$\frac{a^2+ab}{b^2-ab} = \frac{c^2+cd}{d^2-cd},$$

or $$a^2 + ab : b^2 - ab = c^2 + cd : d^2 - cd.$$

(3) If $a : b = c : d$, and a is the *greatest term*, show that $a + d$ is greater than $b + c$.

Since $\frac{a}{b} = \frac{c}{d}$, and $a > c$, (1)

the denominator $b > d$.

From (1), by division, $\frac{a-b}{b} = \frac{c-d}{d}$. (2)

Since $b > d$,

from (2), $a - b > c - d$.

Now, $b + d = b + d$.

Adding, $a + d > b + c$.

288. Ratio of Quantities. To *measure* a quantity of any kind is to find how many times it contains another *known* quantity of the *same kind*, called the **unit of measure.** Thus, if a line contains 5 times the linear unit of measure, one yard, the length of the line is 5 yards.

289. Commensurable Quantities. If two quantities of the *same kind* are so related that a unit of measure can be found which is contained in each of the quantities *an integral number* of times, this unit of measure is a *common measure* of the two quantities, and the two quantities are said to be *commensurable.*

If two commensurable quantities are measured by the same unit, their *ratio* is simply the ratio of the two numbers by which the quantities are expressed. Thus, $\frac{1}{6}$ of a foot is a common measure of $2\frac{1}{2}$ feet and $3\frac{2}{3}$ feet, being contained in the first 15 times and in the second 22 times.

The ratio of $2\frac{1}{2}$ feet to $3\frac{2}{3}$ feet is therefore the ratio of $15 : 22$.

Evidently two quantities *different in kind* can have no ratio.

290. Incommensurable Quantities. The ratio of two quantities of the same kind cannot always be expressed by the ratio of two whole numbers. Thus, the side and the diagonal of a square have no common measure; for if the side is a inches long, the diagonal will be $a\sqrt{2}$ inches long, and no measure can be found which will be contained in each an integral number of times.

Again, the diameter and the circumference of a circle have no common measure, and are therefore incommensurable.

In this case, as there is no common measure of the two quantities, we cannot find their ratio by the method of § 289. We therefore proceed as follows:

Suppose a and b to be two incommensurable quantities of the *same kind*. Divide b into any integral number (n) of equal parts, and suppose one of these parts is contained in a more than m times and less than $m+1$ times. Then the ratio $\frac{a}{b} > \frac{m}{n}$, but $< \frac{m+1}{n}$; that is, the value of $\frac{a}{b}$ lies between $\frac{m}{n}$ and $\frac{m+1}{n}$.

The error, therefore, in taking either of these values for $\frac{a}{b}$ is less than $\frac{1}{n}$. But by increasing n indefinitely, $\frac{1}{n}$ can be made to decrease indefinitely, and to become less than any assigned value, however small, though it cannot be made absolutely equal to zero.

Hence, the ratio of two incommensurable quantities cannot be expressed *exactly* in figures, but it may be expressed *approximately* to any desired degree of accuracy.

Thus, if b represents the side of a square, and a the diagonal,

$$\frac{a}{b} = \sqrt{2}.$$

Now $\sqrt{2} = 1.41421356.....$, a value greater than 1.414213, but less than 1.414214.

If then, a *millionth part* of b is taken as the unit, the value of the ratio $\frac{a}{b}$ lies between $\frac{1414213}{1000000}$ and $\frac{1414214}{1000000}$, and therefore differs from either of these fractions by less than $\frac{1}{1000000}$.

By carrying the decimal farther, a fraction may be found that will differ from the true value of the ratio by less than a *billionth*, a *trillionth*, *or any other assigned value whatever.*

291. The ratio of two incommensurable quantities is an **incommensurable ratio,** and is *a fixed value* toward which its successive approximate values constantly tend as the error is made less and less.

292. Proportion of Quantities. In order for four quantities, A, B, C, D, to be in proportion, A and B must be of the *same kind*, and C and D of the same kind (but C and D need not necessarily be of the same kind as A and B), and in addition the ratio of A to B must be equal to the ratio of C to D.

If this is true, we have the proportion

$$A : B = C : D.$$

When four quantities are in proportion, the numbers by which they are expressed are four abstract numbers in proportion.

293. The laws of § 281, which apply to proportion of numbers, apply also to proportion of quantities, except that alternation will apply only when the four quantities in proportion are *all* of the same kind.

Exercise 107.

If $a : b :: c : d$, prove that:

1. $ma : nb :: mc : nd.$

2. $3a + b : b :: 3c + d : d.$

3. $a + 2b : b :: c + 2d : d.$

4. $a^3 : b^3 :: c^3 : d^3.$

5. $a : a + b :: c : c + d.$

6. $a : a - b :: c : c - d.$

7. $ma + nb : ma - nb :: mc + nd : mc - nd.$

8. $2a + 3b : 3a - 4b :: 2c + 3d : 3c - 4d.$

9. $ma^2 + nc^2 : mb^2 + nd^2 :: a^2 : b^2.$

10. $ma^2 + nab + pb^2 : mc^2 + ncd + pd^2 :: b^2 : d^2.$

If $a : b :: b : c$, prove that:

11. $a + b : b + c :: a : b.$

12. $a^2 + ab : b^2 + bc :: a : c.$

13. $a : c :: (a + b)^2 : (b + c)^2.$

14. When a, b, and c are proportionals, and a the greatest, show that $a + c > 2b$.

15. If $\frac{x-y}{l} = \frac{y-z}{m} = \frac{z-x}{n}$, and x, y, z are unequal, show that $l + m + n = 0$.

16. Find x when $x + 5 : 2x - 3 :: 5x + 1 : 3x - 3$.

17. Find x when $x + a : 2x - b :: 3x + b : 4x - a$.

18. Find x when $\sqrt{x} + \sqrt{b} : \sqrt{x} - \sqrt{b} :: a : b$.

19. Find x and y when $x : 27 :: y : 9$, and $x : 27 :: 2 : x - y$.

20. Find x and y when $x + y + 1 : x + y + 2 :: 6 : 7$, and when $y + 2x : y - 2x :: 12x + 6y - 3 : 6y - 12x - 1$.

21. Find x when $x^2 - 4x + 2 : x^2 - 2x - 1 :: x^2 - 4x : x^2 - 2x - 2$.

22. A railway passenger observes that a train passes him, moving in the opposite direction, in 2 seconds; but moving in the same direction with him, it passes him in 30 seconds. Compare the rates of the two trains.

23. A and B trade with different sums. A gains \$200 and B loses \$50, and now A's stock : B's : : 2 : $\frac{1}{2}$. But, if A had gained \$100 and B lost \$85, their stocks would have been as 15 : $3\frac{1}{4}$. Find the original stock of each.

24. A quantity of milk is increased by watering in the ratio 4 : 5, and then 3 gallons are sold; the remainder is mixed with 3 quarts of water, and is increased in the ratio 6 : 7. How many gallons of milk were there at first?

25. In a mile race between a bicycle and a tricycle their rates were as 5 : 4. The tricycle had half a minute start, but was beaten by 176 yards. Find the rates of each.

26. The time which an express-train takes to travel 180 miles is to that taken by an ordinary train as 9 : 14. The ordinary train loses as much time from stopping as it would take to travel 30 miles; the express-train loses only half as much time as the other by stopping, and travels 15 miles an hour faster. What are their respective rates?

27. A line is divided into two parts in the ratio 2 : 3, and into two parts in the ratio 3 : 4; the distance between the points of section is 2. Find the length of the line.

28. When a, b, c, d, are proportional and unequal, show that no number x can be found such that $a+x$, $b+x$, $c+x$, $d+x$, shall be proportionals.

VARIATION.

294. A quantity which in any particular problem has a fixed value is called a **constant quantity,** or simply a **constant;** a quantity which may change its value is called a **variable quantity,** or simply a **variable.**

Variable numbers, like unknown numbers, are generally represented by x, y, z, etc.; constant numbers, like known numbers, by a, b, c, etc.

295. Two variables may be so related that when a value of one is given, the corresponding value of the other can be found. In this case one variable is said to be a *function* of the other; that is, one variable depends upon the other for its value. Thus, if the rate at which a man walks is known, the distance he walks can be found when the time is given; the distance is in this case a *function* of the time.

296. There is an unlimited number of ways in which two variables may be related. We shall consider in this chapter only a few of these ways.

297. When x and y are so related that their ratio is constant, y is said to vary as x; this is abbreviated thus: $y \propto x$. The sign $\propto$, called the **sign of variation,** is read "varies as." Thus, the area of a triangle with a given base varies as its altitude; for, if the altitude is changed in any ratio, the area will be changed in the same ratio.

In this case, if we represent the constant ratio by m,

$$y : x = m, \text{ or } \frac{y}{x} = m; \therefore y = mx.$$

Again, if y', x' and y'', x'' be two sets of corresponding values of y and x, then

$$y' : x' = y'' : x'';$$

by alternation,

$$y' : y'' = x' : x''.$$

298. When x and y are so related that the ratio of y to $\frac{1}{x}$ is constant, y is said to vary *inversely* as x; this is written $y \propto \frac{1}{x}$. Thus, the time required to do a certain amount of work varies inversely as the number of workmen employed; for, if the number of workmen be doubled, halved, or changed in any other ratio, the time required will be halved, doubled, or changed in the inverse ratio.

In this case, $y : \frac{1}{x} = m$; $\therefore y = \frac{m}{x}$, and $xy = m$; that is, the product xy is constant.

As before, $$y' : \frac{1}{x'} = y'' : \frac{1}{x''},$$

$$x'y' = x''y'',$$

or $$y' : y'' = x'' : x'. \qquad \S\ 280$$

299. If the ratio of $y : xz$ is constant, then y is said to *vary jointly* as x and z.

In this case, $$y = mxz,$$

and $$y' : y'' = x'z' : x''z''.$$

300. If the ratio $y : \frac{x}{z}$ is constant, then y varies *directly* as x and *inversely* as z.

In this case, $$y = \frac{mx}{z},$$

and $$y' : y'' = \frac{mx'}{z'} : \frac{mx''}{z''} = \frac{x'}{z'} : \frac{x''}{z''}.$$

301. Theorems.

I. If $y \propto x$, and $x \propto z$, then $y \propto z$.

For $$y = mx \text{ and } x = nz.$$

$$\therefore y = mnz.$$

$$\therefore y \text{ varies as } z.$$

II. If $y \propto x$, and $z \propto x$, then $(y \pm z) \propto x$.

For $$y = mx \text{ and } z = nx.$$

$$\therefore\ y \pm z = (m \pm n)x.$$

$$\therefore\ y \pm z \text{ varies as } x.$$

III. If $y \propto x$ when z is constant, and $y \propto z$ when x is constant, then $y \propto xz$ when x and z are both variable.

Let x', y', z', and x'', y'', z'', be two sets of corresponding values of the variables.

Let x change from x' to x'', z remaining constant, and let the corresponding value of y be Y.

Then $$y' : Y = x' : x''. \qquad (1)$$

Now let z change from z' to z'', x remaining constant.

Then $$Y : y'' = z' : z''. \qquad (2)$$

Multiply (1) and (2), then,

$$y'Y : y''Y = x'z' : x''z'', \qquad \S\ 285$$

or $$y' : y'' = x'z' : x''z'',$$

or $$y' : x'z' = y'' : x''z''. \qquad \S\ 281, \text{V.}$$

$\therefore$ the ratio $\dfrac{y}{xz}$ is constant, and y varies as xz.

In like manner, it may be shown that if y varies as each of any number of quantities x, z, u, etc., when the rest are unchanged, then when they all change, $y \propto xzu$, etc.

Thus, the area of a rectangle varies as the base when the altitude is constant, and as the altitude when the base is constant, but as the product of the base and altitude when both vary.

The volume of a rectangular solid varies as the length when the width and thickness remain constant; as the width when the length and thickness remain constant; as the thickness when the length and width remain constant; but as the product of length, breadth, and thickness when all three vary.

302. Examples.

(1) If y varies inversely as x, and when $y = 2$ the corresponding value of x is 36, find the corresponding value of x when $y = 9$.

Here $$y = \frac{m}{x}, \text{ or } m = xy$$

$$\therefore\ m = 2 \times 36 = 72.$$

If 9 and 72 are substituted for y and m respectively in

$$y = \frac{m}{x},$$

the result is $$9 = \frac{72}{x}, \text{ or } 9x = 72.$$

$$\therefore\ x = 8.$$

(2) The weight of a sphere of given material varies as its volume, and its volume varies as the cube of its diameter. If a sphere 4 inches in diameter weighs 20 pounds, find the weight of a sphere 5 inches in diameter.

Let W represent the weight,
V represent the volume,
D represent the diameter.

Then $W \propto V$ and $V \propto D^3$.

$\therefore\ W \propto D^3$. § 301, I.

Put $W = mD^3$;

then, since 20 and 4 are corresponding values of W and D,

$$20 = m \times 64.$$

$$\therefore\ m = \frac{20}{64} = \frac{5}{16}.$$

$$\therefore\ W = \tfrac{5}{16} D^3.$$

$\therefore$ when $D = 5$, $W = \frac{5}{16}$ of $125 = 39\frac{1}{16}$.

Exercise 108.

1. If $A \propto B$, and $A = 4$ when $B = 5$, find A when $B = 12$.
2. If $A \propto B$, and when $B = \frac{1}{2}$, $A = \frac{1}{3}$, find A when $B = \frac{1}{3}$.
3. If A vary jointly as B and C, and 3, 4, 5, be simultaneous values of A, B, C, find A when $B = C = 10$.

4. If $A \propto \frac{1}{B}$, and when $A=10$, $B=2$, find the value of B when $A=4$.

5. If $A \propto \frac{B}{C}$, and when $A=6$, $B=4$, and $C=3$, find the value of A when $B=5$ and $C=7$.

6. If the square of X varies as the cube of Y, and $X=3$ when $Y=4$, find the equation between X and Y.

7. If the square of X varies inversely as the cube of Y, and $X=2$ when $Y=3$, find the equation between X and Y.

8. If Z varies as X directly and Y inversely, and if when $Z=2$, $X=3$, and $Y=4$, find the value of Z when $X=15$ and $Y=8$.

9. If $A \propto B+c$ where c is constant, and if $A=2$ when $B=1$, and if $A=5$ when $B=2$, find A when $B=3$.

10. The velocity acquired by a stone falling from rest varies as the time of falling; and the distance fallen varies as the square of the time. If it be found that in 3 seconds a stone has fallen 145 feet, and acquired a velocity of $96\frac{2}{3}$ feet per second, find the velocity and distance at the end of 5 seconds.

11. If a heavier weight draw up a lighter one by means of a string passing over a fixed wheel, the space described in a given time will vary directly as the difference between the weights, and inversely as their sum. If 9 ounces draw 7 ounces through 8 feet in 2 seconds, how high will 12 ounces draw 9 ounces in the same time?

12. The space will vary also as the square of the time. Find the space in Example 11, if the time in the latter case is 3 seconds.

13. Equal volumes of iron and copper are found to weigh 77 and 89 ounces respectively. Find the weight of $10\frac{1}{2}$ feet of round copper rod when 9 inches of iron rod of the same diameter weigh $31\frac{9}{10}$ ounces.

14. The square of the time of a planet's revolution varies as the cube of its distance from the sun. The distances of the Earth and Mercury from the sun being 91 and 35 millions of miles, find in days the time of Mercury's revolution.

15. A spherical iron shell 1 foot in diameter weighs $\frac{91}{216}$ of what it would weigh if solid. Find the thickness of the metal, knowing that the volume of a sphere varies as the cube of its diameter.

16. The volume of a sphere varies as the cube of its diameter. Compare the volume of a sphere 6 inches in diameter with the sum of the volumes of three spheres whose diameters are 3, 4, 5 inches respectively.

17. Two circular gold plates, each an inch thick, the diameters of which are 6 inches and 8 inches respectively, are melted and formed into a single circular plate 1 inch thick. Find its diameter, having given that the area of a circle varies as the square of its diameter.

18. The volume of a pyramid varies jointly as the area of its base and its altitude. A pyramid, the base of which is 9 feet square, and the height of which is 10 feet, is found to contain 10 cubic yards. What must be the height of a pyramid upon a base 3 feet square, in order that it may contain 2 cubic yards?

CHAPTER XXIII.

PROGRESSIONS.

303. A succession of numbers that proceed according to some fixed law is called a **series**; the successive numbers are called the **terms** of the series.

A series that ends at some particular term is a **finite series**; a series that continues without end is an **infinite series.**

304. The number of different forms of series is unlimited; in this chapter we shall consider only Arithmetical Series, Geometrical Series, and Harmonical Series.

Arithmetical Progression.

305. A series is called an **arithmetical series** or an **arithmetical progression** when each succeeding term is obtained by adding to the preceding term a *constant difference.*

The general representative of such a series will be

$$a,\ a + d,\ a + 2d,\ a + 3d\ldots\ldots,$$

in which a is the first term and d the common difference; the series will be *increasing* or *decreasing* according as d is positive or negative.

306. The nth Term. Since each succeeding term of the series is obtained by adding d to the preceding term, the coefficient of d will always be one less than the number of the term, so that the *nth* term is $a + (n - 1)d$.

If the *nth* term is represented by l, we have

$$l = a + (n - 1)d. \qquad \text{I.}$$

307. Sum of the Series. If l denotes the *nth* term, a the first term, n the number of terms, d the common difference, and s the sum of n terms, it is evident that

$$\begin{aligned} s &= a + (a+d) + (a+2d) + \cdots\cdots + (l-d) + l, \text{ or} \\ s &= l + (l-d) + (l-2d) + \cdots\cdots + (a+d) + a. \end{aligned}$$

$$\therefore 2s = (a+l) + (a+l) + (a+l) \quad + \cdots\cdots + (a+l) + (a+l)$$
$$= n(a+l).$$
$$\therefore s = \frac{n}{2}(a+l). \qquad \text{II.}$$

308. From the two equations,

$$l = a + (n-1)d, \qquad \text{I.}$$
$$s = \frac{n}{2}(a+l), \qquad \text{II.}$$

any *two* of the five numbers a, d, l, n, s may be found when the other *three* are given.

(1) Find the sum of ten terms of the series, 2, 5, 8, 11, ...

Here $\quad a = 2,\ d = 3,\ n = 10.$

From I., $\quad l = 2 + 27 = 29.$

Substituting in II., $\quad s = \frac{10}{2}(2 + 29) = 155.$

(2) The first term of an arithmetical series is 3, the last term 31, and the sum of the series 136. Find the series.

From I. and II., $\quad 31 = 3 + (n-1)d, \qquad (1)$

$$136 = \frac{n}{2}(3 + 31). \qquad (2)$$

From (2), $\quad n = 8.$

Substituting in (1), $\quad d = 4.$

The series is 3, 7, 11, 15, 19, 23, 27, 31.

(3) How many terms of the series, 5, 9, 13, ..., must be taken in order that their sum may be 275?

From I., $$l = 5 + (n-1)4.$$
$$\therefore\ l = 4n + 1. \qquad (1)$$

From II., $$275 = \frac{n}{2}(5 + l). \qquad (2)$$

Substituting in (2) the value of l found in (1),

$$275 = \frac{n}{2}(4n + 6),$$

or $$2n^2 + 3n = 275.$$

We now have to solve this quadratic.

Complete the square,

$$16n^2 + (\ ) + 9 = 2209.$$

Extract the root, $$4n + 3 = \pm 47.$$
$$\therefore\ n = 11, \text{ or } -12\tfrac{1}{2}.$$

We use only the positive result.

(4) Find n when d, l, s are given.

From I., $$a = l - (n-1)d.$$

From II., $$a = \frac{2s - ln}{n}.$$

Therefore, $$l - (n-1)d = \frac{2s - ln}{n}.$$
$$\therefore\ ln - dn^2 + dn = 2s - ln.$$
$$\therefore\ dn^2 - (2l + d)n = -2s.$$

This is a quadratic with n for the unknown number.

Complete the square,

$$4d^2n^2 - (\ ) + (2l + d)^2 = (2l + d)^2 - 8ds.$$

Extract the root,

$$2dn - (2l + d) = \pm\sqrt{(2l + d)^2 - 8ds}.$$
$$\therefore\ n = \frac{2l + d \pm \sqrt{(2l + d)^2 - 8ds}}{2d}.$$

NOTE. The table on the following page contains the results of the general solution of all possible problems in arithmetical series, in which three of the numbers a, l, d, n, s are given and two required. The student is advised to work these out, both for the results obtained and for the practice gained in solving literal equations in which the unknown numbers are represented by letters other than x, y, z.

No.	Given.	Required.	Results.
1	$a\ d\ n$	l	$l = a + (n-1)d.$
2	$a\ d\ s$		$l = -\frac{1}{2}d \pm \sqrt{2ds + (a - \frac{1}{2}d)^2}.$
3	$a\ n\ s$		$l = \frac{2s}{n} - a.$
4	$d\ n\ s$		$l = \frac{s}{n} + \frac{(n-1)d}{2}.$
5	$a\ d\ n$	s	$s = \frac{1}{2}n[2a + (n-1)d].$
6	$a\ d\ l$		$s = \frac{l+a}{2} + \frac{l^2 - a^2}{2d}.$
7	$a\ n\ l$		$s = \frac{n}{2}(a + l).$
8	$d\ n\ l$		$s = \frac{1}{2}n[2l - (n-1)d].$
9	$d\ n\ l$	a	$a = l - (n-1)d.$
10	$d\ n\ s$		$a = \frac{s}{n} - \frac{(n-1)d}{2}.$
11	$d\ l\ s$		$a = \frac{1}{2}d \pm \sqrt{(l + \frac{1}{2}d)^2 - 2ds}.$
12	$n\ l\ s$		$a = \frac{2s}{n} - l.$
13	$a\ n\ l$	d	$d = \frac{l-a}{n-1}.$
14	$a\ n\ s$		$d = \frac{2(s - an)}{n(n-1)}.$
15	$a\ l\ s$		$d = \frac{l^2 - a^2}{2s - l - a}.$
16	$n\ l\ s$		$d = \frac{2(nl - s)}{n(n-1)}.$
17	$a\ d\ l$	n	$n = \frac{l-a}{d} + 1.$
18	$a\ d\ s$		$n = \frac{d - 2a \pm \sqrt{(2a-d)^2 + 8ds}}{2d}.$
19	$a\ l\ s$		$n = \frac{2s}{l+a}.$
20	$d\ l\ s$		$n = \frac{2l + d \pm \sqrt{(2l+d)^2 - 8ds}}{2d}.$

309. The **arithmetical mean** between two numbers is the number which stands between them, and makes with them an arithmetical series.

If a and b represent two numbers, and A their arithmetical mean, then, by the definition of an arithmetical series,

$$A - a = b - A.$$

$$\therefore\ A = \frac{a+b}{2}.$$

310. Sometimes it is required to insert several arithmetical means between two numbers.

Insert six arithmetical means between 3 and 17.

Here the whole number of terms is eight; 3 is the first term, and 17 the eighth.

By I., $$17 = 3 + 7d.$$

$$d = 2.$$

The series is 3, [5, 7, 9, 11, 13, 15,] 17,

the terms in brackets being the means required.

311. When the sum of a number of terms in arithmetical progression is given, it is convenient to represent:

Three terms by $x - y,\ \ x,\ \ x + y.$

Four terms by $x - 3y,\ \ x - y,\ \ x + y,\ \ x + 3y.$

The sum of three numbers in arithmetical progression is 36, and the square of the mean exceeds the product of the two extremes by 49. Find the numbers.

Let $x - y$, x, $x + y$ represent the numbers.

Then, adding, $3x = 36.\quad \therefore\ x = 12.$

Putting for x its value, the numbers are

$$12 - y,\quad 12,\quad 12 + y.$$

The value of y is ± 7; and the numbers are

5, 12, 19; or 19, 12, 5.

Exercise 109.

1. Find the thirteenth term of 5, 9, 13 .....
   ninth term of $-3, -1, 1$ .....
   tenth term of $-2, -5, -8$ .....
   eighth term of $a, a+3b, a+6b$ .....
   fifteenth term of $1, \frac{6}{7}, \frac{5}{7}$ .....
   thirteenth term of $-48, -44, -40$ .....

2. The first term of an arithmetical series is 3, the thirteenth term is 55. Find the common difference.

3. Find the arithmetical mean between: (*a.*) 3 and 12; (*b.*) -5 and 17; (*c.*) a^2+ab-b^2 and a^2-ab+b^2.

4. Insert three arithmetical means between 1 and 19; and four means between -4 and 17.

5. The first term of a series is 2, and the common difference $\frac{1}{3}$. What term will be 10?

6. The seventh term of a series, whose common difference is 3, is 11. Find the first term.

7. Find the sum of

$$5+8+11+\cdots\cdots \text{ to ten terms.}$$
$$-4-1+2+\cdots\cdots \text{ to seven terms.}$$
$$a+4a+7a+\cdots\cdots \text{ to } n \text{ terms.}$$
$$\tfrac{2}{3}+\tfrac{7}{15}+\tfrac{4}{15}+\cdots\cdots \text{ to twenty-one terms.}$$
$$1+2\tfrac{2}{3}+4\tfrac{1}{3}+\cdots\cdots \text{ to twenty terms.}$$

8. The sum of six numbers of an arithmetical series is 27, and the first term is 1. Determine the series.

9. How many terms of the series $-5-2+1+\cdots\cdots$ must be taken so that their sum may be 63?

10. The first term is 12, and the sum of ten terms is 10. Find the last term.

11. The arithmetical mean between two numbers is 10, and the mean between the double of the first and the triple of the second is 27. Find the numbers.

12. Find the middle term of eleven terms whose sum is 66.

13. The first term of an arithmetical series is 2, the common difference is 7, and the last term 79. Find the number of terms.

14. The sum of fifteen terms of an arithmetical series is 600, and the common difference is 5. Find the first term.

15. Insert ten arithmetical means between -7 and 114.

16. The sum of three numbers in arithmetical progression is 15, and the sum of their squares is 83. Find the numbers.

Let $x-y$, x, $x+y$ represent the numbers.

17. Arithmetical means are inserted between 5 and 23, so that the sum of the first two is to the sum of the last two as 2 is to 5. How many means are inserted?

18. Find three numbers of an arithmetical series whose sum shall be 21, and the sum of the first and second shall be $\frac{3}{4}$ of the sum of the second and third.

19. Find three numbers whose common difference is 1, such that the product of the second and third exceeds that of the first and second by $\frac{1}{2}$.

20. How many terms of the series 1, 4, 7 ..... must be taken, in order that the sum of the first half may bear to the sum of the second half the ratio 10 : 31?

21. A travels uniformly 20 miles a day; B starts three days later, and travels 8 miles the first day, 12 the second, and so on, in arithmetical progression. In how many days will B overtake A?

22. A number consists of three digits which are in arithmetical progression; and this number divided by the sum of its digits is equal to 26; but if 198 be added to it, the digits in the units' and hundreds' places will be interchanged. Required the number.

23. The sum of the squares of the extremes of four numbers in arithmetical progression is 200, and the sum of the squares of the means is 136. What are the numbers?

24. Show that if any even number of terms of the series 1, 3, 5 ..... be taken, the sum of the first half is to the sum of the second half in the ratio 1 : 3.

25. A and B set out at the same time to meet each other from two places 343 miles apart. Their daily journeys are in arithmetical progression, A's increase being 2 miles each day, and B's decrease being 5 miles each day. On the day at the end of which they met, each travelled exactly 20 miles. Find the duration of the journey.

26. Suppose that a body falls through a space of $16\frac{1}{12}$ feet in the first second of its fall, and in each succeeding second $32\frac{1}{6}$ more than in the next preceding one. How far will a body fall in 20 seconds?

27. The sum of five numbers in arithmetical progression is 45, and the product of the first and fifth is $\frac{5}{8}$ of the product of the second and fourth. Find the numbers.

28. If a full car descending an incline draw up an empty one at the rate of $1\frac{1}{2}$ feet the first second, $4\frac{1}{2}$ feet the next second, $7\frac{1}{2}$ feet the third, and so on, how long will it take to descend an incline 150 feet in length? What part of the distance will the car have descended in the first half of the time?

Geometrical Progression.

312. A series is called a **geometrical series** or a **geometrical progression** when each succeeding term is obtained by multiplying the preceding term by a *constant multiplier*.

The general representative of such a series will be

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4 \ldots\ldots,$$

in which a is the first term and r the constant multiplier or ratio.

The terms increase or decrease in numerical magnitude according as r is numerically greater than or numerically less than unity.

313. The nth Term. Since the exponent of r increases by one for each succeeding term after the first, the exponent will always be one less than the number of the term, so that the nth term is ar^{n-1}.

If the nth term is represented by l, we have

$$l = ar^{n-1}. \qquad \text{I.}$$

314. Sum of the Series. If l represents the nth term, a the first term, n the number of terms, r the common ratio, and s the sum of n terms, then

$$s = a + ar + ar^2 + \cdots\cdots ar^{n-1}.$$

Multiply by r,

$$rs = ar + ar^2 + ar^3 + \cdots\cdots ar^{n-1} + ar^n.$$

Subtracting the first equation from the second,

$$rs - s = ar^n - a,$$

or $$(r-1)\,s = a(r^n - 1).$$

$$\therefore\ s = \frac{a(r^n - 1)}{r - 1}. \qquad \text{II.}$$

Since $ar^{n-1} = l$, it follows that $ar^n = rl$, and II. may be written

$$s = \frac{rl - a}{r - 1}. \qquad \text{III.}$$

315. From the two equations I. and II., or the two equations I. and III., any *two* of the five numbers a, r, l, n, s, may be found when the other *three* are given.

(1) The first term of a geometrical series is 3, the last term 192, and the sum of the series 381. Find the number of terms and the ratio.

From I. and III., $$192 = 3r^{n-1}, \qquad (1)$$

$$381 = \frac{192r - 3}{r - 1}. \qquad (2)$$

From (2), $$r = 2.$$

Substituting in (1), $$2^{n-1} = 64 = 2^6.$$

$$\therefore n = 7.$$

The series is 3, 6, 12, 24, 48, 96, 192.

(2) Find l when r, n, s are given.

From I., $$a = \frac{l}{r^{n-1}}.$$

Substituting in III., $$s = \frac{rl - \dfrac{l}{r^{n-1}}}{r - 1},$$

$$(r - 1)s = \frac{(r^n - 1)}{r^{n-1}}l.$$

$$\therefore l = \frac{(r - 1)r^{n-1}s}{r^n - 1}.$$

NOTE. The table on page 305 contains the results of all possible problems in geometrical series in which three of the numbers a, r, l, n, s, are given and the other two required, with the exception of those in which n is required; these last require the use of logarithms with which the student is supposed to be not yet acquainted.

No.	Given.	Required.	Results.
1	$a\ r\ n$	l	$l = ar^{n-1}.$
2	$a\ r\ s$		$l = \frac{a + (r-1)s}{r}.$
3	$a\ n\ s$		$l(s-l)^{n-1} - a(s-a)^{n-1} = 0.$
4	$r\ n\ s$		$l = \frac{(r-1)sr^{n-1}}{r^n - 1}.$
5	$a\ r\ n$	s	$s = \frac{a(r^n - 1)}{r - 1}.$
6	$a\ r\ l$		$s = \frac{rl - a}{r - 1}.$
7	$a\ n\ l$		$s = \frac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}.$
8	$r\ n\ l$		$s = \frac{lr^n - l}{r^n - r^{n-1}}.$
9	$r\ n\ l$	a	$a = \frac{l}{r^{n-1}}.$
10	$r\ n\ s$		$a = \frac{(r-1)s}{r^n - 1}.$
11	$r\ l\ s$		$a = rl - (r-1)s.$
12	$n\ l\ s$		$a(s-a)^{n-1} - l(s-l)^{n-1} = 0.$
13	$a\ n\ l$	r	$r = \sqrt[n-1]{\frac{l}{a}}.$
14	$a\ n\ s$		$r^n - \frac{s}{a}r + \frac{s-a}{a} = 0.$
15	$a\ l\ s$		$r = \frac{s-a}{s-l}.$
16	$n\ l\ s$		$r^n - \frac{s}{s-l}r^{n-1} + \frac{l}{s-l} = 0.$

316. The **geometrical mean** between two numbers is the number which stands between them, and makes with them a geometrical series,

If a and b denote two numbers, and G their geometrical mean, then, by the definition of a geometrical series,

$$\frac{G}{a}=\frac{b}{G}.$$

$$\therefore\ G=\sqrt{ab}.$$

317. Sometimes it is required to insert several geometrical means between two numbers.

Insert three geometrical means between 3 and 48.

Here the whole number of terms is five; 3 is the first term, and 48 the fifth.

By I.,
$$48=3r^4,$$
$$r^4=16,$$
$$r=\pm 2.$$

The series is either of the following:

$$3,\ [\ 6,\ 12,\ 24,]\ 48;$$
$$3,\ [-6,\ 12,\ -24,]\ 48.$$

The terms in brackets are the means required.

318. Infinite Geometrical Series. When r is less than 1, the successive terms become numerically smaller and smaller; by taking n large enough we can make the nth term, ar^{n-1}, as small as we please, although we cannot make it absolutely zero.

The sum of n terms, $\dfrac{rl-a}{r-1}$, may be written $\dfrac{a}{1-r}-\dfrac{rl}{1-r}$; this sum differs from $\dfrac{a}{1-r}$ by the fraction $\dfrac{rl}{1-r}$; by taking enough terms we can make l, and consequently this difference, as small as we please; the greater the number of terms taken, the nearer does their sum approach $\dfrac{a}{1-r}$. Hence $\dfrac{a}{1-r}$ is called the *sum* of an infinite number of terms of the series.

(1) Find the sum of the infinite series

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\dots\dots$$

Here, $a=1$, $r=-\frac{1}{2}$.

The sum of the series is $\frac{1}{1+\frac{1}{2}}$ or $\frac{2}{3}$.

We find for the sum of n terms $\frac{2}{3}-\frac{2}{3}\left(-\frac{1}{2}\right)^n$; this sum evidently approaches $\frac{2}{3}$ as n is increased.

(2) Find the value of the recurring decimal 0.12135135.....

Consider first the part that recurs; this may be written $\frac{135}{100000}+\frac{135}{100000000}+\dots\dots$, and the sum of this series is $\frac{\frac{135}{100000}}{1-\frac{1}{1000}}$, which reduces to $\frac{1}{740}$. Adding 0.12, the part that does not recur, we obtain for the value of the decimal $\frac{449}{3700}$.

Exercise 110.

1. Find the seventh term of 2, 6, 18.....
   sixth term of 3, 6, 12.....
   ninth term of 6, 3, $1\frac{1}{2}$.....
   eighth term of 1, -2, 4.....
   twelfth term of x^3, x^4, x^5.....
   fifth term of $4a$, $-6ma^2$, $9m^2a^3$.....

2. Find the geometrical mean between $18x^3y$ and $30xy^3z$

3. Find the ratio when the first and third terms are 5 and 80 respectively.

4. Insert two geometrical means between 8 and 125; and three between 14 and 224.

5. If $a = 2$ and $r = 3$, which term is equal to 162?

6. The fifth term of a geometrical series is 48, and the ratio 2. Find the first and seventh terms.

7. Find the sum of

$$3 + 6 + 12 + \cdots\cdots \text{ to eight terms.}$$
$$1 - 3 + 9 - \cdots\cdots \text{ to seven terms.}$$
$$8 + 4 + 2 + \cdots\cdots \text{ to ten terms.}$$
$$0.1 + 0.5 + 2.5 + \cdots\cdots \text{ to seven terms.}$$
$$m - \frac{m}{4} + \frac{m}{16} - \cdots\cdots \text{ to five terms.}$$

8. The population of a city increases in four years from 10,000 to 14,641. What is the rate of increase?

9. The sum of four numbers in geometrical progression is 200, and the first term is 5. Find the ratio.

10. Find the sum of eight terms of a series whose last term is 1, and fifth term $\frac{1}{8}$.

11. In an odd number of terms, show that the product of the first and last will be equal to the square of the middle term.

12. The product of four terms of a geometrical series is 4, and the fourth term is 4. Determine the series.

13. If from a line one-third be cut off, then one-third of the remainder, and so on, what fraction of the whole will remain when this has been done five times?

14. Of three numbers in geometrical progression, the sum of the first and second exceeds the third by 3, and the sum of the first and third exceeds the second by 21. What are the numbers?

15. Find two numbers whose sum is $3\frac{1}{4}$ and geometrical mean $1\frac{1}{2}$.

16. A glass of wine is taken from a decanter that holds ten glasses, and a glass of water poured in. After this is done five times, what part of the contents is wine?

17. There are four numbers such that the sum of the first and the last is 11, and the sum of the others is 10. The first three of these four numbers are in arithmetical progression, and the last three are in geometrical progression. Find the numbers.

18. Find three numbers in geometrical progression such that their sum is 13 and the sum of their squares is 91.

19. The difference between two numbers is 48, and the arithmetical mean exceeds the geometrical by 18. Find the numbers.

20. There are four numbers in geometrical progression, the second of which is less than the fourth by 24, and the sum of the extremes is to the sum of the means as 7 to 3. Find the numbers.

21. A number consists of three digits in geometrical progression. The sum of the digits is 13; and if 792 be added to the number, the digits in the units' and hundreds' places will be interchanged. Find the number.

22. Find the sum of each of the infinite series:

$4 + 2 + 1 + \cdots\cdots$ $\qquad$ $2 - 1\frac{1}{3} + \frac{8}{9} - \cdots\cdots$

$\frac{1}{2} + \frac{1}{3} + \frac{2}{9} + \cdots\cdots$ $\qquad$ $0.1 + 0.01 + 0.001 + \cdots\cdots$

$\frac{1}{4} - \frac{1}{16} + \frac{1}{64} - \cdots\cdots$ $\qquad$ $0.868686\ldots\ldots$

$1 - \frac{2}{5} + \frac{4}{25} - \cdots\cdots$ $\qquad$ $0.54444\ldots\ldots$

$\frac{1}{5} + \frac{1}{15} + \frac{1}{45} + \cdots\cdots$ $\qquad$ $0.83636\ldots\ldots$

Harmonical Progression.

319. A series is called a **harmonical series,** or a **harmonical progression,** when the reciprocals of its terms form an arithmetical series.

The general representative of such a series will be

$$\frac{1}{a},\ \frac{1}{a+d},\ \frac{1}{a+2d},\ \ldots\ldots,\ \frac{1}{a+(n-1)d}.$$

Questions relating to harmonical series are best solved by writing the reciprocals of its terms, and thus forming an arithmetical series.

320. If a and b denote two numbers, and H their harmonical mean, then, by the definition of a harmonical series,

$$\frac{1}{H}-\frac{1}{a}=\frac{1}{b}-\frac{1}{H}.$$

$$\therefore\ \frac{2}{H}=\frac{1}{a}+\frac{1}{b}=\frac{a+b}{ab}.$$

$$\therefore\ H=\frac{2ab}{a+b}.$$

321. Sometimes it is required to insert several harmonical means between two numbers.

Insert three harmonical means between 3 and 18.

Find the three arithmetical means between $\frac{1}{3}$ and $\frac{1}{18}$.

These are found to be $\frac{19}{72}, \frac{14}{72}, \frac{9}{72}$; therefore, the harmonical means are $\frac{72}{19}, \frac{72}{14}, \frac{72}{9}$; or $3\frac{15}{19}$, $5\frac{1}{7}$, 8.

322. Since, §§ 309, 316, 320,

$$A=\frac{a+b}{2},\ \ G=\sqrt{ab},\ \text{ and }\ H=\frac{2ab}{a+b},$$

we have

$$A:G=G:H.$$

Exercise 111.

1. Insert four harmonical means between 2 and 12.

2. Find two numbers whose difference is 8 and the harmonical mean between them $1\frac{4}{5}$.

3. Find the seventh term of the harmonical series 3, $3\frac{3}{7}$, 4 .....

4. Continue to two terms each way the harmonical series, two consecutive terms of which are 15, 16.

5. The first two terms of a harmonical series are 5 and 6. Which term will equal 30?

6. The fifth and ninth terms of a harmonical series are 8 and 12. Find the first four terms.

7. The difference between the arithmetical and harmonical means between two numbers is $1\frac{4}{5}$, and one of the numbers is four times the other. Find the numbers.

8. Find the arithmetical, geometrical, and harmonical means between two numbers a and b; and show that the geometrical mean is a mean proportional between the arithmetical and harmonical means. Also, arrange these means in order of magnitude.

9. The arithmetical mean between two numbers exceeds the geometrical by 13, and the geometrical exceeds the harmonical by 12. What are the numbers?

10. The sum of three terms of a harmonical series is 11, and the sum of their squares is 49. Find the numbers.

11. When a, b, c are in harmonical progression, show that $a : c :: a - b : b - c$.

CHAPTER XXIV.

INDETERMINATE COEFFICIENTS.

323. Convergent and Divergent Series. By performing the indicated division, we obtain from the fraction $\frac{1}{1-x}$ the infinite series $1+x+x^2+x^3+\cdots\cdots$. This series, however, is not equal to the fraction for all values of x.

324. If x is numerically less than 1, the series is equal to the fraction. In this case we can obtain an approximate value for the sum of the series by taking the sum of a number of terms; the greater the number of terms taken, the nearer will this approximate sum approach the value of the fraction. The approximate sum will never be exactly equal to the fraction, however great the number of terms taken; but by taking enough terms, it can be made to differ from the fraction as little as we please.

Thus, if $x=\frac{1}{2}$, the value of the fraction is 2, and the series is

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\cdots$$

The sum of four terms of this series is $1\frac{7}{8}$; the sum of five terms, $1\frac{15}{16}$; the sum of six terms, $1\frac{31}{32}$; and so on. The successive approximate sums approach, but never reach, the finite value 2.

325. An infinite series is said to be **convergent** when the sum of the terms, as the number of terms is indefinitely increased, *approaches indefinitely some fixed finite value;* this finite value is called the **sum of the series.**

326. In the series $1 + x + x^2 + x^3 + \cdots\cdots$ suppose x numerically greater than 1. In this case, the greater the number of terms taken, the greater will their sum be; by taking enough terms, we can make their sum as large as we please. The fraction, on the other hand, has a definite value. Hence, when x is numerically greater than 1, the series is *not* equal to the fraction.

Thus, if $x = 2$, the value of the fraction is -1, and the series is

$$1 + 2 + 4 + 8 + \cdots\cdots$$

The greater the number of terms taken, the larger the sum. Evidently the fraction and the series are not equal.

327. In the same series suppose $x = 1$. In this case the fraction is $\frac{1}{1-1} = \frac{1}{0}$, and the series $1 + 1 + 1 + 1 + \cdots\cdots$ The more terms we take, the greater will the sum of the series be, and the sum of the series does *not* approach a *fixed finite value.*

If x, however, is not exactly 1, but is a little less than 1, the value of the fraction $\frac{1}{1-x}$ will be very great, and the fraction will be equal to the series.

Suppose $x = -1$. In this case the fraction is $\frac{1}{1+1} = \frac{1}{2}$, and the series $1 - 1 + 1 - 1 + \cdots\cdots$ If we take an even number of terms, their sum is 0; if an odd number, their sum is 1. Hence the fraction is *not* equal to the series.

328. A series is said to be **divergent** when the sum of the terms, as the number of terms is indefinitely increased, either increases without end, or oscillates in value *without approaching any fixed finite value.*

No reasoning can be based on a divergent series; hence, in using an infinite series it is necessary to make such restrictions as will cause the series to be convergent. Thus, we can use the infinite series $1 + x + x^2 + x^3 + \cdots\cdots$ when, and only when, x lies between $+1$ and -1.

329. A series, $ax + bx^2 + cx^3 + dx^4 + \cdots\cdots$, in which the coefficients $a, b, c, d \ldots\ldots$ are finite, may, by taking x sufficiently small, be made less than any assigned value.

For if q is any assigned value, and k the greatest of the coefficients a, b, c, ..., then

$$ax + bx^2 + cx^3 + \cdots\cdots < kx + kx^2 + kx^3 + \cdots\cdots$$

But $$kx + kx^2 + kx^3 + \cdots\cdots = \frac{kx}{1-x}$$

(as is evident by dividing kx by $1 - x$).

$\therefore\ ax + bx^2 + cx^3 + \cdots\cdots < \dfrac{kx}{1-x}$, if x is taken less than 1.

Hence, if $\dfrac{kx}{1-x}$ be taken less than q,

that is, if $x < \dfrac{q}{q+k}$,

then $ax + bx^2 + cx^3 + \cdots\cdots$ will be less than q.

330. Theorem of Indeterminate Coefficients. *If two series, arranged by powers of x, are equal for all values of x that make both series convergent, the corresponding coefficients are equal each to each.*

For, if $A + Bx + Cx^2 + \cdots\cdots = A' + B'x + C'x^2 + \cdots\cdots$,

by transposition,

$$A - A' = (B' - B)x + (C' - C)x^2 + \cdots\cdots$$

Now by taking x sufficiently small, the right side of this equation can be made *less* than any assigned value whatever, and therefore less than $A - A'$, if $A - A'$ has any value whatever. Hence $A - A'$ cannot have any value.

$\therefore\ A - A' = 0$ or $A = A'$.

Hence, $\quad Bx + Cx^2 + Dx^3 + \cdots\cdots = B'x + C'x^2 + D'x^3 + \cdots\cdots$

or $\quad (B - B')x = (C' - C)x^2 + (D' - D)x^3 + \cdots\cdots;$

by dividing by x,

$$B - B' = (C' - C)x + (D' - D)x^2 + \cdots\cdots;$$

and, by the same proof as for $A - A'$,

$$B - B' = 0 \text{ or } B = B'.$$

In like manner,

$$C = C',\ D = D', \text{ and so on.}$$

Hence, the equation

$$A + Bx + Cx^2 + \cdots\cdots = A' + B'x + C'x^2 + \cdots\cdots,$$

if true for all finite values of x, is an **identical equation**; that is, *the coefficients of like powers of x are the same.*

Expand $\dfrac{2+3x}{1+x+x^2}$ in ascending powers of x.

Assume $\quad \dfrac{2+3x}{1+x+x^2} = A + Bx + Cx^2 + Dx^3 + \cdots\cdots;$

then, by clearing of fractions,

$$\begin{aligned} 2 + 3x = A + Bx &+ Cx^2 + Dx^3 + \cdots\cdots \\ + Ax &+ Bx^2 + Cx^3 + \cdots\cdots \\ &+ Ax^2 + Bx^3 + \cdots\cdots \end{aligned}$$

$$\therefore\ 2 + 3x = A + (B + A)x + (C + B + A)x^2 + (D + C + B)x^3 + \cdots\cdots$$

$$\therefore\ A = 2,\ B + A = 3,\ C + B + A = 0,\ D + C + B = 0;$$

whence $\quad B = 1,\ C = -3,\ D = 2;$ and so on.

$$\therefore\ \frac{2+3x}{1+x+x^2} = 2 + x - 3x^2 + 2x^3 + \cdots\cdots$$

The series is of course equal to the fraction for only such values of x as make the series convergent.

NOTE. In employing the method of Indeterminate Coefficients, the form of the given expression must determine what powers of the variable x must be assumed. It is necessary and sufficient that the assumed equation, when simplified, shall have in the right member all the powers of x that are found in the left member.

If any powers of x occur in the *right* member that are not in the *left* member, the coefficients of these powers in the right member will vanish, so that in this case the method still applies; but if any powers of x occur in the *left* member that are not in the *right* member, then the coefficients of these powers of x must be put equal to 0 in equating the coefficients of like powers of x; and this leads to absurd results. Thus, if it were assumed that

$$\frac{2+3x}{1+x+x^2} = Ax + Bx^2 + Cx^3 + \cdots\cdots,$$

there would be in the simplified equation no term on the right corresponding to 2 on the left; so that, in equating the coefficients of like powers of x, 2, which is $2x^0$, would have to be put equal to $0x^0$; that is, $2 = 0$, an absurdity.

Exercise 112.

Expand to four terms in ascending powers of x:

1. $\dfrac{1}{2-3x}$. **2.** $\dfrac{1+x}{2+3x}$. **3.** $\dfrac{3-2x}{4-3x}$. **4.** $\dfrac{1-x}{1-x+x^2}$.

5. $\dfrac{1}{1-2x+3x^2}$. **6.** $\dfrac{5-2x}{1+3x-x^2}$. **7.** $\dfrac{4x-6x^2}{1-2x+3x^2}$.

331. Partial Fractions. To resolve a fraction into *partial fractions* is to express it as the sum of a number of fractions of which the respective denominators are the factors of the denominator of the given fraction. This process is the reverse of the process of *adding* fractions which have different denominators.

Resolution into partial fractions may be easily accomplished by the use of **indeterminate coefficients.**

In decomposing a given fraction into its simplest partial

fractions, it is important to determine what form the assumed fractions must have.

Since the given fraction is the *sum* of the required partial fractions, each assumed denominator must be a factor of the given denominator; moreover, all the factors of the given denominator must be taken as denominators of the assumed fractions.

Since the required partial fractions are to be in their simplest form incapable of further decomposition, the numerator of each required fraction must be assumed with reference to this condition. Thus, if the denominator is x^n or $(x \pm a)^n$, the assumed fraction must be of the form $\frac{A}{x^n}$ or $\frac{A}{(x \pm a)^n}$; for if it had the form $\frac{Ax+B}{x^n}$ or $\frac{Ax+B}{(x \pm a)^n}$, it could be decomposed into two fractions, and the partial fractions would not be in the simplest form possible.

When all the monomial factors, and all the binomial factors, of the form $x \pm a$ have been removed from the denominator of the given expression, there may remain quadratic factors which cannot be further resolved; and the numerators corresponding to these quadratic factors may each contain the first power of x, so that the assumed fractions must have either the form $\frac{Ax+B}{x^2 \pm ax + b}$, or the form $\frac{Ax+B}{x^2+b}$.

(1) Resolve $\frac{3}{x^3+1}$ into partial fractions.

Since $x^3+1=(x+1)(x^2-x+1)$, the denominators will be $x+1$ and x^2-x+1.

Assume
$$\frac{3}{x^3+1}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1};$$

then
$$\begin{aligned}3 &= A(x^2-x+1)+(Bx+C)(x+1)\\ &= (A+B)x^2+(B+C-A)x+(A+C);\end{aligned}$$

whence, $A + C = 3,\ B + C - A = 0,\ A + B = 0,$

and $A = 1,\ B = -1,\ C = 2.$

Therefore, $$\frac{3}{x^3+1} = \frac{1}{x+1} - \frac{x-2}{x^2-x+1}.$$

(2) Resolve $\dfrac{4x^3 - x^2 - 3x - 2}{x^2(x+1)^2}$ into partial fractions.

The denominators may be x, x^2, $x+1$, $(x+1)^2$.

Assume $$\frac{4x^3-x^2-3x-2}{x^2(x+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}.$$

$$\therefore\ 4x^3 - x^2 - 3x - 2 = Ax(x+1)^2 + B(x+1)^2 + Cx^2(x+1) + Dx^2$$
$$= (A+C)x^3 + (2A + B + C + D)x^2 + (A + 2B)x + B;$$

whence,
$$A + C = 4,$$
$$2A + B + C + D = -1,$$
$$A + 2B = -3,$$
$$B = -2;$$

or $B = -2,\ A = 1,\ C = 3,\ D = -4.$

Therefore, $$\frac{4x^3-x^2-3x-2}{x^2(x+1)^2} = \frac{1}{x} - \frac{2}{x^2} + \frac{3}{x+1} - \frac{4}{(x+1)^2}.$$

Exercise 113.

Resolve into partial fractions:

1. $\dfrac{7x+1}{(x+4)(x-5)}$.
2. $\dfrac{6}{(x+3)(x+4)}$.
3. $\dfrac{5x-1}{(2x-1)(x-5)}$.
4. $\dfrac{x-2}{x^2-3x-10}$.
5. $\dfrac{3}{x^3-1}$.
6. $\dfrac{x^2-x-3}{x(x^2-4)}$.
7. $\dfrac{3x^2-4}{x^2(x+5)}$.
8. $\dfrac{7x^2-x}{(x-1)^2(x+2)}$.
9. $\dfrac{2x^2-7x+1}{x^3-1}$.

CHAPTER XXV.

BINOMIAL THEOREM.

332. Binomial Theorem, Positive Integral Exponent. By successive multiplications we obtain the following identities:

$$(a+b)^2 = a^2 + 2ab + b^2;$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3;$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

The expressions on the right may be written in a form better adapted to show the law of their formation:

$$(a+b)^2 = a^2 + 2ab + \frac{2\cdot 1}{1\cdot 2}b^2;$$
$$(a+b)^3 = a^3 + 3a^2b + \frac{3\cdot 2}{1\cdot 2}ab^2 + \frac{3\cdot 2\cdot 1}{1\cdot 2\cdot 3}b^3;$$
$$(a+b)^4 = a^4 + 4a^3b + \frac{4\cdot 3}{1\cdot 2}a^2b^2 + \frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3}ab^3 + \frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4}b^4.$$

NOTE. The dot between the Arabic figures means the same as the sign $\times$.

333. Let n represent the exponent of $(a+b)$ in any one of these identities; then, in the expressions on the right, we observe that the following laws hold true:

I. The number of terms is $n+1$.

II. The first term is a^n, and the exponent of a is one less in each succeeding term.

The first power of b occurs in the second term, the second power in the third term, and the exponent of b is one greater in each succeeding term.

The sum of the exponents of a and b in any term is n.

III. The coefficient of the first term is 1; of the second term, n; of the third term, $\frac{n(n-1)}{1 \cdot 2}$; and so on.

334. Consider the coefficient of any term; the number of factors in the numerator is the same as the number of factors in the denominator, and the number of factors in each is the same as the exponent of b in that term; this exponent is one less than the number of the term.

335. Proof of the Theorem. To show that the laws of § 333 hold true when the exponent is *any* positive integer:

We know that the laws hold for the fourth power; suppose, for the moment, that they hold for the kth power.

We shall then have

$$(a+b)^k = a^k + ka^{k-1}b + \frac{k(k-1)}{1 \cdot 2}a^{k-2}b^2 + \frac{k(k-1)(k-2)}{1 \cdot 2 \cdot 3}a^{k-3}b^3 + \cdots\cdots \quad (1)$$

Multiply both members of (1) by $a+b$; the result is

$$(a+b)^{k+1} = a^{k+1} + (k+1)a^kb + \frac{(k+1)k}{1 \cdot 2}a^{k-1}b^2 + \frac{(k+1)k(k-1)}{1 \cdot 2 \cdot 3}a^{k-2}b^3 + \cdots\cdots \quad (2)$$

In (1) put $k+1$ for k; this gives

$$(a+b)^{k+1} = a^{k+1} + (k+1)a^kb + \frac{(k+1)(k+1-1)}{1 \cdot 2}a^{k-1}b^2 + \frac{(k+1)(k+1-1)(k+1-2)}{1 \cdot 2 \cdot 3}a^{k-2}b^3 + \cdots\cdots$$

$$= a^{k+1} + (k+1)a^kb + \frac{(k+1)k}{1 \cdot 2}a^{k-1}b^2 + \frac{(k+1)k(k-1)}{1 \cdot 2 \cdot 3}a^{k-2}b^3 + \cdots\cdots \quad (3)$$

Equation (3) is seen to be the same as equation (2).

Hence (1) holds when we put $k+1$ for k; that is, if the laws of § 333 hold for the kth power, they must hold for the $(k+1)$th power.

But the laws hold for the fourth power; therefore they must hold for the fifth power.

Holding for the fifth power, they must hold for the sixth power; and so on for any positive integral power.

Therefore they must hold for the nth power, if n is a positive integer; and we have

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots\cdots \qquad (\mathbf{A})$$

NOTE. The above proof is an example of a proof by *mathematical induction.*

336. This formula is known as the **binomial theorem.**

The expression on the right is known as the **expansion** of $(a+b)^n$; this expansion is a *finite series* when n is a positive integer. That the series is finite may be seen as follows:

In writing the successive coefficients we shall finally arrive at a coefficient which contains the factor $n-n$; the corresponding term will vanish. The coefficients of all the succeeding terms likewise contain the factor $n-n$, and therefore all these terms will vanish.

337. If a and b are interchanged, the identity (**A**) may be written

$$(a+b)^n = (b+a)^n = b^n + nb^{n-1}a + \frac{n(n-1)}{1\cdot 2}b^{n-2}a^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}b^{n-3}a^3 + \cdots\cdots$$

This last expansion is the expansion of (**A**) written in reverse order. Comparing the two expansions, we see that: the coefficient of the last term is the same as the coefficient of the first term; the coefficient of the last term but one is the same as the coefficient of the first term but one; and so on.

In general, the coefficient of the rth term from the end is the same as the coefficient of the rth term from the beginning. In writing an expansion by the binomial theorem, after arriving at the middle term, we can shorten the work by observing that the remaining coefficients are those already found, taken in reverse order.

338. If b is negative, the terms which involve even powers of b will be positive, and those which involve odd powers of b negative. Hence,

$$(a-b)^n = a^n - na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots\cdots \qquad \text{(B)}$$

Also, putting 1 for a and x for b, in (**A**) and (**B**),

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots\cdots \qquad \text{(C)}$$

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{1\cdot 2}x^2 - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots\cdots \qquad \text{(D)}$$

339. Examples.

(1) Expand $(1+2x)^5$.

In (C) put $2x$ for x and 5 for n. The result is

$$(1+2x)^5 = 1 + 5(2x) + \frac{5\cdot 4}{1\cdot 2}4x^2 + \frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3}8x^3$$

$$+ \frac{5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4}16x^4 + \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4\cdot 5}32x^5$$

$$= 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5.$$

(2) Expand to three terms $\left(\frac{1}{x} - \frac{2x^2}{3}\right)^6$.

Put a for $\frac{1}{x}$, and b for $\frac{2x^2}{3}$; then, by (B),

$$(a-b)^6 = a^6 - 6a^5b + 15a^4b^2 - \cdots\cdots$$

Replacing a and b by their values,

$$\left(\frac{1}{x} - \frac{2x^2}{3}\right)^6 = \left(\frac{1}{x}\right)^6 - 6\left(\frac{1}{x}\right)^5\left(\frac{2x^2}{3}\right) + 15\left(\frac{1}{x}\right)^4\left(\frac{2x^2}{3}\right)^2 - \cdots\cdots$$

$$= \frac{1}{x^6} - \frac{4}{x^3} + \frac{20}{3} - \cdots\cdots$$

340. Any Required Term. From (A) it is evident (§ 335) that the $(r+1)$th term of the expansion of $(a+b)^n$ is

$$\frac{n(n-1)(n-2)\ldots\ldots \text{ to } r \text{ factors}}{1\times 2\times 3 \ldots\ldots r}a^{n-r}b^r.$$

NOTE. In finding the coefficient of the $(r+1)$th term, write down the series of factors $1\times 2\times 3 \ldots\ldots r$ for the denominator of the coefficient, then write over this series the factors $n(n-1)(n-2)\cdot$ etc., writing just as many factors in the numerator as there are in the denominator.

The $(r+1)$th term in the expansion of $(a-b)^n$ is the same as the above if r is even, and the negative of the above if r is odd.

Find the eighth term of $\left(4-\frac{x^2}{2}\right)^{10}$.

Here $a=4,\ b=\frac{x^2}{2},\ n=10,\ r=7.$

The term required is $\frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}(4)^3\left(-\frac{x^2}{2}\right)^7$.

which reduces to $-60x^{14}$.

341. A trinomial may be expanded by the binomial theorem as follows:

Expand $(1+2x-x^2)^3$.

Put $2x-x^2=z$;

then $(1+z)^3=1+3z+3z^2+z^3.$

Replace z with $2x-x^2$.

$$\begin{aligned}\therefore\ (1+2x-x^2)^3 &= 1+3(2x-x^2)+3(2x-x^2)^2+(2x-x^2)^3\\ &= 1+6x+9x^2-4x^3-9x^4+6x^5-x^6.\end{aligned}$$

Exercise 114.

1. $(1+2x)^5$.
2. $(x-3)^8$.
3. $(2x-3y)^4$.
4. $(2-x)^3$.
5. $\left(1-\frac{3y}{4}\right)^5$.
6. $\left(1-\frac{x}{3}\right)^9$.
7. Find the fourth term of $(2x-5y)^{12}$.
8. Find the seventh term of $\left(\frac{x}{2}+\frac{y}{3}\right)^{10}$.
9. Find the twelfth term of $(a^2-ax)^{15}$.
10. Find the eighth term of $(5x^2y-2xy^2)^9$.
11. Find the middle term of $\left(\frac{x}{y}+\frac{y}{x}\right)^8$.
12. Find the middle term of $\left(\frac{x}{y}-\frac{y}{x}\right)^{10}$.
13. Find the two middle terms of $\left(\frac{x}{y}-\frac{y}{x}\right)^7$.

14. Find the rth term of $(2a+x)^n$.

15. Find the rth term from the end of $(2a+x)^n$.

16. Find the $(r+4)$th term of $(a+x)^n$.

17. Find the middle term of $(a+x)^{2n}$.

18. Expand $(2a+x)^{12}$, and find the sum of the terms if $a=1$, $x=-2$.

Expand:

19. $(\sqrt{a}+\sqrt{b})^5$. **24.** $(\sqrt[3]{m^2}+\sqrt{x^3})^3$. **29.** $(\sqrt{a}-2\sqrt{b})^5$.

20. $(2a^2-\frac{1}{2}\sqrt{a})^6$. **25.** $(2\sqrt[5]{x^4}-\frac{1}{2}y^2)^4$. **30.** $\left(\frac{2x^2}{y}-\sqrt[3]{y^2}\right)^6$.

21. $\left(\sqrt{ab}-\frac{c}{2\sqrt{b}}\right)^5$. **26.** $\left(\frac{a^2}{2c}-\frac{\sqrt{c}}{3}\right)^5$. **31.** $\left(a^2b-\frac{\sqrt{b}}{2a}\right)^4$.

22. $\left(\frac{a}{b}\sqrt{\frac{c}{d}}-\sqrt{\frac{d^2}{c^2}}\right)^3$. **27.** $(a^{\frac{n}{2}}-a^{\frac{n}{2}})^4$. **32.** $\left(\frac{2a}{b^2}-\frac{1}{3}b\sqrt{a}\right)^4$.

23. $\left(\sqrt{\frac{a}{bc}}-\frac{\sqrt{c}}{3ab}\right)^3$. **28.** $\left(\frac{\sqrt{a}}{2\sqrt[3]{b^2}}-3\sqrt{b}\right)^3$. **33.** $\left(\frac{a\sqrt{a}}{\sqrt[6]{b^5}}-\frac{\sqrt[4]{b}}{2a}\right)^3$.

342. Binomial Theorem, Any Exponent. We have seen (§ 338) that when n is a positive integer we have the identity

$$(1+x)^n=1+nx+\frac{n(n-1)}{1\cdot 2}x^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3+\cdots\cdots$$

We proceed to the case of fractional and negative exponents.

I. Suppose n is a positive fraction, $\frac{p}{q}$. We may assume that

$$(1+x)^p = (A + Bx + Cx^2 + Dx^3 + \cdots\cdots)^q, \tag{1}$$

provided x be so taken that the series

$$A + Bx + Cx^2 + Dx^3 + \cdots\cdots$$

is convergent (§ 325).

That this assumption is allowable may be seen as follows:

Expand both members of (1). We obtain

$$1 + px + \frac{p(p-1)}{1 \cdot 2}x^2 + \frac{p(p-1)(p-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots\cdots,$$

and $$A^q + qA^{q-1}Bx + \left(\frac{q(q-1)}{1 \cdot 2}A^{q-2}B^2 + qA^{q-1}C\right)x^2 + \cdots\cdots$$

In the first k coefficients of the second series there enter only the first k of the coefficients A, B, C, D, ... If, then, we equate the coefficients of corresponding terms in the two series (§ 330) as far as the kth term, *we shall have just k equations to find k unknown numbers* A, B, C, D, ... Hence the assumption made in (1) is allowable.

Comparing the two first terms and the two second terms, we obtain

$$A^q = 1, \quad \therefore\ A = 1;$$

$$qA^{q-1}B = p, \text{ or } qB = p, \quad \therefore\ B = \frac{p}{q}.$$

Extracting the qth root of both members of (1), we have

$$(1+x)^{\frac{p}{q}} = 1 + \frac{p}{q}x + Cx^2 + Dx^3 + \cdots\cdots, \tag{2}$$

where x is so taken that the series on the right is convergent.

II. Suppose n is a negative number, integral or fractional. Let $n = -m$, so that m is positive; then

$$(1+x)^n = (1+x)^{-m} = \frac{1}{(1+x)^m}.$$

From (2), whether m is integral or fractional, we may assume

$$\frac{1}{(1+x)^m} = \frac{1}{1 + mx + cx^2 + dx^3 + \cdots\cdots}.$$

By actual division this gives an equation in the form

$$(1+x)^{-m} = 1 - mx + Cx^2 + Dx^3 + \cdots\cdots \qquad (3)$$

343. It appears from (2) and (3) § 342 that whether n be integral or fractional, positive or negative, we may assume

$$(1+x)^n = 1 + nx + Cx^2 + Dx^3 + \cdots\cdots,$$

provided the series on the right is convergent.

Squaring both members,

$$\begin{aligned}(1+2x+x^2)^n = 1 + 2nx + 2Cx^2 &+ 2Dx^3 + \cdots\cdots \qquad (1)\\ + n^2x^2 &+ 2nCx^3 + \cdots\cdots\end{aligned}$$

Also, since

$$(1+y)^n = 1 + ny + Cy^2 + Dy^3 + \cdots\cdots,$$

we have, putting $2x + x^2$ for y,

$$\begin{aligned}(1+2x+x^2)^n &= 1 + n(2x+x^2) + C(2x+x^2)^2\\ &\qquad + D(2x+x^2)^3 + \cdots\cdots\\ &= 1 + 2nx + nx^2 + 4Cx^3 + \cdots\cdots\\ &\qquad\qquad + 4Cx^2 + 8Dx^3 + \cdots\cdots \qquad (2)\end{aligned}$$

Comparing corresponding coefficients in (1) and (2),

$$n + 4C = 2C + n^2,$$
$$4C + 8D = 2D + 2nC.$$

$$\therefore\ 2C = n^2 - n, \text{ and } C = \frac{n(n-1)}{1\cdot 2},$$

$$3D = (n-2)C, \text{ and } D = \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3};$$

and so on.

Hence, whether n be integral or fractional, positive or negative, we have

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots\cdots,$$

if x is so taken that the series on the right is convergent.

The series obtained will be an infinite series unless n is a positive integer (§ 336).

344. If x is negative,

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{1\cdot 2}x^2 - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots\cdots$$

Also, if $x < a$,

$$\begin{aligned}(a+x)^n &= a^n\left(1+\frac{x}{a}\right)^n \\ &= a^n\left(1 + n\frac{x}{a} + \frac{n(n-1)}{1\cdot 2}\frac{x^2}{a^2} + \cdots\cdots\right) \\ &= a^n + na^{n-1}x + \frac{n(n-1)}{1\cdot 2}a^{n-2}x^2 + \cdots\cdots;\end{aligned}$$

if $x > a$,

$$\begin{aligned}(a+x)^n = (x+a)^n &= x^n\left(1+\frac{a}{x}\right)^n \\ &= x^n\left(1 + n\frac{a}{x} + \frac{n(n-1)}{1\cdot 2}\frac{a^2}{x^2} + \cdots\cdots\right) \\ &= x^n + nax^{n-1} + \frac{n(n-1)}{1\cdot 2}a^2x^{n-2} + \cdots\cdots\end{aligned}$$

345. Examples.

(1) Expand $(1+x)^{\frac{1}{3}}$.

$$\begin{aligned}(1+x)^{\frac{1}{3}} &= 1 + \tfrac{1}{3}x + \frac{\frac{1}{3}(\frac{1}{3}-1)}{1\cdot 2}x^2 + \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{1\cdot 2\cdot 3}x^3 + \cdots\cdots \\ &= 1 + \tfrac{1}{3}x - \frac{2}{3\cdot 6}x^2 + \frac{2\cdot 5}{3\cdot 6\cdot 9}x^3 - \cdots\cdots\end{aligned}$$

if x is so taken that the series is convergent.

(2) Expand $(1+x)^{-\frac{1}{3}}$.

$$(1+x)^{-\frac{1}{3}} = 1 + (-\tfrac{1}{3})x + \frac{-\frac{1}{3}\cdot -\frac{4}{3}}{1\cdot 2}x^2 + \frac{-\frac{1}{3}\cdot -\frac{4}{3}\cdot -\frac{7}{3}}{1\cdot 2\cdot 3}x^3 + \cdots\cdots$$

$$= 1 - \tfrac{1}{3}x + \frac{1\cdot 4}{3\cdot 6}x^2 - \frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}x^3 + \cdots\cdots$$

if x is so taken that the series is convergent.

(3) Expand $\dfrac{1}{\sqrt[4]{1-x}}$.

$$\frac{1}{\sqrt[4]{1-x}} = (1-x)^{-\frac{1}{4}}$$

$$= 1 - (-\tfrac{1}{4})x + \frac{-\frac{1}{4}\cdot -\frac{5}{4}}{1\cdot 2}x^2 - \frac{-\frac{1}{4}\cdot -\frac{5}{4}\cdot -\frac{9}{4}}{1\cdot 2\cdot 3}x^3 + \cdots\cdots$$

$$= 1 + \tfrac{1}{4}x + \frac{1\cdot 5}{4\cdot 8}x^2 + \frac{1\cdot 5\cdot 9}{4\cdot 8\cdot 12}x^3 + \cdots\cdots$$

if x is so taken that the series is convergent.

A root may often be extracted by means of an expansion.

(4) Extract the cube root of 344 to six decimal places.

$$344 = 343\left(1 + \frac{1}{343}\right) = 7^3\left(1 + \frac{1}{343}\right).$$

$$\therefore\ \sqrt[3]{344} = 7\left(1 + \frac{1}{343}\right)^{\frac{1}{3}},$$

$$= 7\left(1 + \frac{1}{3}\left(\frac{1}{343}\right) + \frac{\frac{1}{3}(\frac{1}{3}-1)}{1\cdot 2}\left(\frac{1}{343}\right)^2 + \cdots\cdots\right)$$

$$= 7(1 + 0.000971817 - 0.000000944)$$

$$= 7.006796.$$

(5) Find the eighth term of $\left(x - \dfrac{3}{4\sqrt{x}}\right)^{-\frac{1}{2}}$.

Here $\quad a = x,\ \ b = \dfrac{3}{4\sqrt{x}} = \dfrac{3}{4x^{\frac{1}{2}}},\ \ n = -\dfrac{1}{2},\ \ r = 7.$

The term is $\quad \dfrac{-\frac{1}{2}\cdot -\frac{3}{2}\cdot -\frac{5}{2}\cdot -\frac{7}{2}\cdot -\frac{9}{2}\cdot -\frac{11}{2}\cdot -\frac{13}{2}}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^{-\frac{15}{2}}\left(-\dfrac{3}{4x^{\frac{1}{2}}}\right)^7$

or $\quad \dfrac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13\cdot 3^7}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12\cdot 14\cdot 4^7\cdot x^{11}}.$

Exercise 115.

Expand to four terms:

1. $(1+x)^{\frac{1}{2}}$.
2. $(1+x)^{\frac{2}{3}}$.
3. $(a+x)^{\frac{3}{4}}$.
4. $(1-x)^{-4}$.
5. $(a^2-x^2)^{\frac{5}{2}}$.
6. $(x^2+xy)^{-\frac{3}{2}}$.
7. $(2x-3y)^{-\frac{1}{4}}$.
8. $\sqrt[5]{1-5x}$.
9. $\dfrac{1}{\sqrt{(4a^2-3ax)^3}}$.
10. $\sqrt[6]{\dfrac{1}{(1-3y)^5}}$.
11. $(1+x+x^2)^{\frac{2}{3}}$.
12. $(1-x+x^2)^{\frac{3}{2}}$.
13. Find the rth term of $(a+x)^{\frac{1}{2}}$.
14. Find the rth term of $(a-x)^{-3}$.
15. Find $\sqrt{65}$ to five decimal places.
16. Find $\sqrt[3]{1\frac{1}{30}}$ to five decimal places.
17. Find $\sqrt[7]{129}$ to six decimal places.
18. Expand $(1-2x+3x^2)^{-\frac{2}{5}}$ to four terms.
19. Find the coefficient of x^4 in the expansion of $\dfrac{(1+2x)^2}{(1+3x)^3}$.
20. By means of the expansion of $(1+x)^{\frac{1}{2}}$ show that the limit of the series

$$1+\frac{1}{2}-\frac{1}{2\times 2^2}+\frac{1\times 3}{2\times 3\times 2^3}-\frac{1\times 3\times 5}{2\times 3\times 4\times 2^4}+\cdots\text{ is }\sqrt{2}.$$

CHAPTER XXVI.

COMMON LOGARITHMS.

346. If the natural numbers are regarded as powers of ten, the exponents of the powers are the **Common or Briggs Logarithms** of the numbers. If A and B denote natural numbers, a and b their logarithms, then

$$10^a = A,\ 10^b = B;$$

or, in logarithmic form,

$$\log A = a,\ \log B = b.$$

347. The logarithm of a product is found by adding the logarithms of its factors.

For $$A \times B = 10^a \times 10^b = 10^{a+b}.$$

Therefore, $$\log (A \times B) = a + b = \log A + \log B.$$

348. The logarithm of a quotient is found by subtracting the logarithm of the divisor from that of the dividend.

For $$\frac{A}{B} = \frac{10^a}{10^b} = 10^{a-b}.$$

Therefore, $$\log \frac{A}{B} = a - b = \log A - \log B.$$

349. The logarithm of a power is found by multiplying the logarithm of the number by the exponent of the power.

For $$A^n = (10^a)^n = 10^{an}.$$

Therefore, $$\log A^n = an = n \log A.$$

350. The logarithm of the root of a number is found by dividing the logarithm of the number by the index of the root.

For $$\sqrt[n]{A} = \sqrt[n]{10^a} = 10^{\frac{a}{n}}.$$

Therefore, $$\log \sqrt[n]{A} = \frac{a}{n} = \frac{\log A}{n}.$$

351. The logarithms of 1, 10, 100, etc., and of 0.1, 0.01, 0.001, etc., are integral numbers. The logarithms of all other numbers are fractions.

$$\begin{array}{llll}
\text{Since} & 10^0 = 1, & & 10^{-1}\left(=\tfrac{1}{10}\right) = 0.1, \\
& 10^1 = 10, & & 10^{-2}\left(=\tfrac{1}{100}\right) = 0.01, \\
& 10^2 = 100, & & 10^{-3}\left(=\tfrac{1}{1000}\right) = 0.001, \\
\text{therefore} & \log 1 = 0, & & \log 0.1 = -1, \\
& \log 10 = 1, & & \log 0.01 = -2, \\
& \log 100 = 2, & & \log 0.001 = -3.
\end{array}$$

Also, it is evident that the common logarithms of all numbers between

$$\begin{array}{rl}
1 \text{ and } 10 \text{ will be} & 0 + \text{a fraction}, \\
10 \text{ and } 100 \text{ will be} & 1 + \text{a fraction}, \\
100 \text{ and } 1000 \text{ will be} & 2 + \text{a fraction}, \\
1 \text{ and } 0.1 \text{ will be} & -1 + \text{a fraction}, \\
0.1 \text{ and } 0.01 \text{ will be} & -2 + \text{a fraction}, \\
0.01 \text{ and } 0.001 \text{ will be} & -3 + \text{a fraction}.
\end{array}$$

352. If the number is less than 1, the logarithm is negative (§ 351), but is written in such a form that the *fractional part* is always *positive.*

353. Every logarithm, therefore, consists of two parts: a positive or negative integral number, which is called the **characteristic,** and a *positive* proper fraction, which is called

the **mantissa**. Thus, in the logarithm 3.5218, the integral number 3 is the characteristic, and the fraction .5218 the mantissa. In the logarithm $0.7825 - 2$, which is sometimes written $\bar{2}.7825$, the integral number -2 is the characteristic, and the fraction 0.7825 is the mantissa.

354. If the logarithm has a negative characteristic, it is customary to change its form by adding 10, or a multiple of 10, to the characteristic, and then indicating the subtraction of the same number from the result. Thus, the logarithm $\bar{2}.7825$ is changed to $8.7825 - 10$ by adding 10 to the characteristic and writing -10 after the result. The logarithm $\overline{13}.9273$ is changed to $7.9273 - 20$ by adding 20 to the characteristic and writing -20 after the result.

355. The following rules are derived from § 351:

Rule 1. If the number is *greater than* 1, make the *characteristic* of the logarithm *one unit less* than the number of figures on the left of the decimal point.

Rule 2. If the number is *less than* 1, make the characteristic of the logarithm *negative*, and *one unit more* than the number of zeros between the decimal point and the first significant figure of the given number.

Rule 3. If the characteristic of a given logarithm is *positive*, make the number of figures in the integral part of the corresponding number *one more* than the number of units in the characteristic.

Rule 4. If the characteristic is *negative*, make the number of zeros between the decimal point and the first significant figure of the corresponding number *one less* than the number of units in the characteristic.

Thus, the characteristic of log 7849.27 is 3; the characteristic of log 0.037 is $-2 = 8.0000 - 10$. If the characteristic

is 4, the corresponding number has five figures in its integral part. If the characteristic is -3, that is, $7.0000-10$, the corresponding fraction has two zeros between the decimal point and the first significant figure.

356. The *mantissa* of the common logarithm of any integral number, or decimal fraction, depends only upon the digits of the number, and is unchanged so long as the *sequence of the digits* remains the same.

For changing the position of the decimal point in a number is equivalent to multiplying or dividing the number by a power of 10. Its common logarithm, therefore, will be increased or diminished by the *exponent* of that power of 10; and since this exponent is *integral*, the *mantissa*, or decimal part of the logarithm, will be unaffected.

$$\begin{aligned} \text{Thus,}\quad 27196 &= 10^{4.4345}, & 2.7196 &= 10^{0.4345},\\ 2719.6 &= 10^{3.4345}, & 0.27196 &= 10^{9.4345-10},\\ 27.196 &= 10^{1.4345}, & 0.0027196 &= 10^{7.4345-10}. \end{aligned}$$

One advantage of using the number *ten* as the base of a system of logarithms consists in the fact that the *mantissa* depends only on the *sequence of digits*, and the *characteristic* on the *position of the decimal point.*

357. In simplifying the logarithm of a root the equal positive and negative numbers to be added to the logarithm should be such that the resulting negative number, when divided by the index of the root, gives a quotient of -10.

Thus, if the $\log 0.002^{\frac{1}{3}} = \frac{1}{3}$ of $(7.3010-10)$, the expression $\frac{1}{3}$ of $(7.3010-10)$ may be put in the form $\frac{1}{3}$ of $(27.3010-30)$, which is $9.1003-10$, since the addition of 20 to the 7, and of -20 to the -10, produces no change in the *value* of the logarithm.

Exercise 116.

Given: $\log 2 = 0.3010$; $\log 3 = 0.4771$; $\log 5 = 0.6990$; $\log 7 = 0.8451$.

Find the common logarithms of the following numbers by resolving the numbers into factors, and taking the sum of the logarithms of the factors.

1. $\log 35$.	**5.** $\log 12$.	**9.** $\log 0.05$.	**13.** $\log 1.75$.
2. $\log 9$.	**6.** $\log 60$.	**10.** $\log 12.5$.	**14.** $\log 105$.
3. $\log 8$.	**7.** $\log 75$.	**11.** $\log 1.25$.	**15.** $\log 0.0105$.
4. $\log 49$.	**8.** $\log 7.5$.	**12.** $\log 37.5$.	**16.** $\log 1.05$.

Find the common logarithms of the following:

17. 7^4.	**20.** $5^{\frac{1}{2}}$.	**23.** $2^{\frac{3}{4}}$.	**26.** $3^{\frac{9}{11}}$.	**29.** $5^{\frac{3}{4}}$.
18. 3^8.	**21.** $3^{\frac{1}{8}}$.	**24.** $5^{\frac{2}{3}}$.	**27.** $7^{\frac{7}{2}}$.	**30.** $7^{\frac{11}{7}}$.
19. 7^3.	**22.** $7^{\frac{1}{5}}$.	**25.** $3^{\frac{3}{7}}$.	**28.** $3^{\frac{4}{3}}$.	**31.** $21^{\frac{7}{8}}$.

358. The logarithm of the reciprocal of a number is called the **cologarithm** of the number.

If A denote any number, then

$$\text{colog}\, A = \log \frac{1}{A} = \log 1 - \log A \; (\S\, 348) = -\log A \; (\S\, 351).$$

Hence, the cologarithm of a number is equal to the logarithm of the number with the minus sign prefixed, which sign affects the entire logarithm, both characteristic and mantissa.

In order to avoid a negative mantissa in the cologarithm, it is customary to substitute for $-\log A$ its equivalent $(10 - \log A) - 10$.

Hence, the cologarithm of a number is found by subtracting the logarithm of the number from 10, and then annexing -10 to the remainder.

The best way to perform the subtraction is to begin on the left and subtract each figure of $\log A$ from 9 until we reach the last significant figure, which must be subtracted from 10.

If $\log A$ is greater in absolute value than 10 and less than 20, then in order to avoid a negative mantissa, it is necessary to write $-\log A$ in the form $(20 - \log A) - 20$. So that, in this case, $\operatorname{colog} A$ is found by subtracting $\log A$ from 20, and then annexing -20 to the remainder.

(1) Find the cologarithm of 4007.

$$\begin{array}{lrl} & & 10 \qquad -10 \\ \text{Given:} & \log 4007 = & \underline{3.6028\qquad\quad} \\ \text{Therefore} & \operatorname{colog} 4007 = & 6.3972 - 10 \end{array}$$

(2) Find the cologarithm of 103992000000.

$$\begin{array}{lrl} & & 20 \qquad -20 \\ \text{Given:} & \log 103992000000 = & \underline{11.0170\qquad\quad} \\ \text{Therefore,} & \operatorname{colog} 103992000000 = & 8.9830 - 20 \end{array}$$

If the characteristic of $\log A$ is negative, then the subtrahend, -10 or -20, will vanish in finding the value of $\operatorname{colog} A$.

(3) Find the cologarithm of 0.004007.

$$\begin{array}{lrl} & & 10 \qquad -10 \\ \text{Given:} & \log 0.004007 = & \underline{7.6028 - 10} \\ \text{Therefore,} & \operatorname{colog} 0.004007 = & 2.3972 \end{array}$$

By using cologarithms the inconvenience of subtracting the logarithm of a divisor is avoided. For dividing by a number is equivalent to multiplying by its reciprocal. Hence, instead of subtracting the logarithm of a divisor, its cologarithm may be added.

(1) Find the logarithm of $\frac{5}{0.002}$.

$$\log\frac{5}{0.002} = \log 5 + \operatorname{colog} 0.002.$$

$$\begin{aligned} \log 5 &= 0.6990 \\ \operatorname{colog} 0.002 &= \underline{2.6990} \\ \log \text{quotient} &= 3.3980 \end{aligned}$$

(2) Find the logarithm of $\frac{0.07}{2^3}$.

$$\log\frac{0.07}{2^3} = \log 0.07 + \operatorname{colog} 2^3.$$

$$\begin{aligned} \log 0.07 &= 8.8451 - 10 \\ \operatorname{colog} 2^3 = (10 - 3\log 2) - 10 &= \underline{9.0970 - 10} \\ \log \text{quotient} &= 7.9421 - 10 \end{aligned}$$

Exercise 117.

Given: $\log 2 = 0.3010$; $\log 3 = 0.4771$; $\log 5 = 0.6990$; $\log 7 = 0.8451$; $\log 11 = 1.0414$.

Find the logarithms of the following quotients:

1. $\frac{2}{5}$.	**7.** $\frac{5}{3}$.	**13.** $\frac{0.05}{3}$.	**19.** $\frac{0.05}{0.003}$.	**25.** $\frac{0.02^2}{3^3}$.
2. $\frac{2}{7}$.	**8.** $\frac{5}{2}$.	**14.** $\frac{0.005}{2}$.	**20.** $\frac{0.007}{0.02}$.	**26.** $\frac{3^3}{0.02^2}$.
3. $\frac{3}{5}$.	**9.** $\frac{7}{3}$.	**15.** $\frac{0.07}{5}$.	**21.** $\frac{0.02}{0.007}$.	**27.** $\frac{7^3}{0.02^2}$.
4. $\frac{3}{7}$.	**10.** $\frac{7}{2}$.	**16.** $\frac{5}{0.07}$.	**22.** $\frac{0.005}{0.07}$.	**28.** $\frac{0.07^3}{0.003^3}$.
5. $\frac{5}{7}$.	**11.** $\frac{3}{2}$.	**17.** $\frac{3}{0.007}$.	**23.** $\frac{0.03}{7}$.	**29.** $\frac{0.005^2}{7^3}$.
6. $\frac{7}{5}$.	**12.** $\frac{7}{0.5}$.	**18.** $\frac{0.003}{7}$.	**24.** $\frac{0.0007}{0.2}$.	**30.** $\frac{7^3}{0.005^2}$.

359. Tables. A table of *four-place* common logarithms is given on pages 340 and 341, which contains the common logarithms of all numbers under 1000, *the decimal point and characteristic being omitted.* The logarithms of single digits, 1, 8, etc., will be found at 10, 80, etc.

Tables containing logarithms of more places can be procured, but this table will serve for many practical uses, and will enable the student to use tables of five-place, seven-place, and ten-place logarithms, in work that requires greater accuracy.

In working with a four-place table, the numbers corresponding to the logarithms, that is, the **antilogarithms**, as they are called, may be carried to *four significant digits.*

360. To find the Logarithm of a Number in this Table.

(1) Suppose it required to find the logarithm of 65.7. In the column headed "N" look for the first two significant figures, and at the top of the table for the third significant figure. In the line with 65, and in the column headed 7, is seen 8176. To this number prefix the characteristic and insert the decimal point. Thus,

$$\log 65.7 = 1.8176.$$

(2) Suppose it is required to find the logarithm of 20347. In the line with 20, and in the column headed 3, is seen 3075; also in the line with 20, and in the 4 column, is seen 3096, and the difference between these two is 21. The difference between 20300 and 20400 is 100, and the difference between 20300 and 20347 is 47. Hence, $\frac{47}{100}$ of $21 = 10$, nearly, must be added to 3075; that is,

$$\log 20347 = 4.3085.$$

(3) Suppose it is required to find the logarithm of 0.0005076. In the line with 50, and in the 7 column, is seen 7050; in the 8 column, 7059: the difference is 9. The

difference between 5070 and 5080 is 10, and the difference between 5070 and 5076 is 6. Hence, $\frac{6}{10}$ of $9 = 5$ must be added to 7050; that is,

$$\log 0.0005076 = 6.7055 - 10.$$

361. To find a Number when its Logarithm is given.

(1) Suppose it is required to find the number of which the logarithm is 1.9736.

Look for 9736 in the table. In the column headed "N," and in the line with 9736, is seen 94, and at the head of the column in which 9736 stands is seen 1. Therefore, write 941, and insert the decimal point as the characteristic directs; that is, the number required is 94.1.

(2) Suppose it is required to find the number of which the logarithm is 3.7936.

Look for 7936 in the table. It cannot be found, but the two adjacent mantissas between which it lies are seen to be 7931 and 7938; their difference is 7, and the difference between 7931 and 7936 is 5. Therefore, $\frac{5}{7}$ of the difference between the numbers corresponding to the mantissas, 7931 and 7938, must be added to the number corresponding to the mantissa 7931.

The number corresponding to the mantissa 7938 is 6220.

The number corresponding to the mantissa 7931 is 6210.

The difference between these numbers is 10,

and $$6210 + \tfrac{5}{7} \text{ of } 10 = 6217.$$

Therefore, the number required is 6217.

(3) Suppose it is required to find the number of which the logarithm is $7.3882 - 10$.

Look for 3882 in the table. It cannot be found, but the two adjacent mantissas between which it lies are seen to be 3874 and 3892; the difference between the two mantissas is 18, and the difference between 3874 and the given mantissa 3882 is 8.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1071	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3647	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8290	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

The number corresponding to the mantissa 3892 is 2450.
The number corresponding to the mantissa 3874 is 2440.
The difference between these numbers is 10,

and $$2440 + \tfrac{8}{18} \text{ of } 10 = 2444.$$

Therefore, the number required is 0.002444.

Exercise 118.

Find from the table the logarithms of:

1. 999. **4.** 90801. **7.** 0.00987. **10.** 7.0699.

2. 9901. **5.** 10001. **8.** 0.87701. **11.** 0.0897.

3. 5406. **6.** 10010. **9.** 1.0001. **12.** 99.778.

Find antilogarithms to the following common logarithms:

13. 2.5310. **15.** 9.8800 – 10. **17.** 7.0216 – 10.

14. 1.9484. **16.** 0.2787. **18.** 8.6580 – 10.

362. Examples.

(1) Find the product of $908.4 \times 0.05392 \times 2.117$.

$$\begin{array}{lrl} \log & 908.4 = & 2.9583 \\ \log & 0.05392 = & 8.7318 - 10 \\ \log & 2.117 = & 0.3257 \\ \hline & & 2.0158 = \log 103.7. \end{array}$$

When any of the factors are *negative*, find their logarithms without regard to the signs; write – after the logarithm that corresponds to a negative number. If the number of logarithms so marked is *odd*, the product is *negative*; if *even*, the product is *positive*.

(2) Find the quotient of $\dfrac{-8.3709 \times 834.637}{7308.946}$.

$$\begin{array}{lrll} \log & 8.3709 = & 0.9227 & - \\ \log & 834.637 = & 2.9215 & + \\ \text{colog} & 7308.946 = & 6.1362 - 10 & + \\ \hline & & 9.9804 - 10 = \log -0.9558. & \end{array}$$

(3) Find the cube of 0.0497.

$$\begin{array}{lr} & \log 0.0497 = 8.6964 - 10 \\ \text{Multiply by 3,} & 3 \\ \hline & 6.0892 - 10 = \log 0.0001228. \end{array}$$

(4) Find the fourth root of 0.00862.

$$\begin{array}{ll} & \log 0.00862 = 7.9355 - 10 \\ \text{Add } 30 - 30, & 30 \qquad - 30 \\ \text{Divide by 4,} & 4)\underline{37.9355 - 40} \\ & 9.4839 - 10 = \log 0.3047. \end{array}$$

(5) Find the value of $\sqrt[5]{\dfrac{3.1416 \times 4771.21 \times 2.7183^{\frac{1}{2}}}{30.103^4 \times 0.4343^{\frac{1}{2}} \times 69.897^4}}$.

$$\begin{array}{rrl} \log & 3.1416 = 0.4971 & = 0.4971 \\ \log & 4771.21 = 3.6786 & = 3.6786 \\ \frac{1}{2}\log & 2.7183 = 0.4343 \div 2 & = 0.2172 \\ 4 \text{ colog} & 30.103 = 4(8.5214 - 10) & = 4.0856 - 10 \\ \frac{1}{2}\text{ colog} & 0.4343 = 0.3622 \div 2 & = 0.1811 \\ 4 \text{ colog} & 69.897 = 4(8.1555 - 10) & = 2.6220 - 10 \\ \hline & & 11.2816 - 20 \\ & & 30 \qquad - 30 \\ \hline & & 5)\underline{41.2816 - 50} \\ & & 8.2563 - 10 \\ & & = \log 0.01804. \end{array}$$

363. An **exponential equation**, that is, an equation in which the exponent involves the unknown number, is easily solved by Logarithms.

Ex. Find the value of x in $81^x = 10$.

$$81^x = 10.$$

$$\therefore \log (81^x) = \log 10,$$

$$x \log 81 = \log 10,$$

$$x = \frac{\log 10}{\log 81} = \frac{1.0000}{1.9085} = 0.524.$$

Exercise 119.

Find by logarithms the following:

1. 948.76×0.043875.

2. 3.4097×0.0087634.

3. 830.75×0.0003769.

4. 8.4395×0.98274.

5. $7564 \times (-0.003764)$.

6. $3.7648 \times (-0.083497)$.

7. $-5.840359 \times (-0.00178)$.

8. -8945.07×73.846.

9. $\dfrac{70654}{54013}$.

10. $\dfrac{58706}{93078}$.

11. $\dfrac{8.32165}{0.07891}$.

12. $\dfrac{65039}{90761}$.

13. $\dfrac{7.652}{-0.06875}$.

14. $\dfrac{0.07654}{83.947 \times 0.8395}$.

15. $\dfrac{7564 \times 0.07643}{8093 \times 0.09817}$.

16. $\dfrac{89 \times 753 \times 0.0097}{36709 \times 0.08497}$.

17. $\dfrac{413 \times 8.17 \times 3182}{915 \times 728 \times 2.315}$.

18. $\dfrac{212 \times (-6.12) \times (-2008)}{365 \times (-531) \times 2.576}$.

19. 6.05^3.

20. 1.051^7.

21. 1.1768^5.

22. 1.3178^{10}.

23. 0.78765^6.

24. 0.691^9.

25. $(\frac{73}{61})^{11}$.

26. $(\frac{14}{51})^7$.

27. $(10\frac{2}{3})^4$.

28. $(1\frac{7}{9})^8$.

29. $(\frac{951}{823})^6$.

30. $(7\frac{6}{11})^{0.38}$.

31. $(3\frac{27}{31})^{4.17}$.

32. $(1\frac{2}{11})^{3.2}$.

33. $(8\frac{3}{4})^{2.3}$.

34. $(5\frac{31}{37})^{0.375}$.

35. $7^{\frac{1}{3}}$.

36. $11^{\frac{1}{5}}$.

37. $783^{\frac{1}{3}}$.

38. $8379^{\frac{1}{10}}$.

39. $906.80^{\frac{1}{4}}$.

40. $8.1904^{\frac{1}{5}}$.

41. $0.17643^{\frac{5}{6}}$.

42. $2.5637^{\frac{3}{11}}$.

43. $(\frac{431}{788})^{\frac{1}{2}}$.

44. $(\frac{71}{43406})^{\frac{4}{7}}$.

45. $(9\frac{21}{43})^{\frac{1}{5}}$.

46. $(11\frac{21}{71})^{\frac{4}{5}}$.

47. $5^x = 20$.

48. $(1.3)^x = 2.1$.

49. $(0.9)^x = \frac{1}{2}$.

50. $\sqrt[5]{\dfrac{0.0075433^2 \times 78.343 \times 8172.4^{\frac{1}{3}} \times 0.00052}{64285.^{\frac{1}{3}} \times 154.27^4 \times 0.001 \times 586.79^{\frac{1}{2}}}}.$

51. $\sqrt[5]{\dfrac{15.832^3 \times 5793.6^{\frac{1}{3}} \times 0.78426}{0.000327^{\frac{1}{3}} \times 768.94^2 \times 3015.3 \times 0.007^{\frac{1}{2}}}}.$

52. $\sqrt[5]{\dfrac{7.1895 \times 4764.2^2 \times 0.00326^5}{0.00048953 \times 457^3 \times 5764.4^2}}.$

53. $\sqrt[5]{\dfrac{3.1416 \times 4771.21 \times 2.7183^{\frac{1}{2}}}{30.103^4 \times 0.4343^{\frac{1}{2}} \times 69.897^4}}.$

54. $\sqrt[7]{\dfrac{0.03271^2 \times 53.429 \times 0.77542^3}{32.769 \times 0.000371^4}}.$

55. $\sqrt[3]{\dfrac{732.056^2 \times 0.0003572^4 \times 89793}{42.2798^3 \times 3.4574 \times 0.0026518^5}}.$

56. $\sqrt[3]{\dfrac{7932 \times 0.00657 \times 0.80464}{0.03274 \times 0.6428}}.$

57. $\sqrt[3]{\dfrac{7.1206 \times \sqrt{0.13274} \times 0.057389}{\sqrt{0.43468} \times 17.385 \times \sqrt{0.0096372}}}.$

58. $\left\{\dfrac{3.075526^2 \times 5771.2^{\frac{1}{2}} \times 0.0036984^{\frac{1}{5}} \times 7.74}{72258 \times 327.93^3 \times 86.97^5}\right\}^{\frac{3}{5}}.$

NOTE. It is *assumed* in this chapter that the index laws which have been established for commensurable exponents hold good for incommensurable exponents. For the proof see Wentworth's College Algebra, § 264, page 216.

Any positive number, except 1, may be selected as the base; and to the base selected there corresponds a *system of logarithms.*

CHAPTER XXVII.

INTEREST AND ANNUITIES.

364. Simple Interest.

If the principal is represented by P,
the interest on \$1 for one year by r,
the amount of \$1 for one year by R,
the number of years by n,
the amount of P for n years by A,

Then $R = 1 + r$.

Simple interest on P for a year $= Pr$,
Amount of P for a year $= PR$,
Simple interest on P for n years $= Pnr$,
Amount of P for n years $= P(1 + nr)$,
that is, $A = P(1 + nr)$.

365. When any three of the quantities A, P, n, r are given, the fourth may be found.

Required the rate when \$500 in 4 years at simple interest amounts to \$610.

r is required, A, P, n are given.

$$A = P(1 + nr),$$

or

$$A = P + Pnr.$$

$$\therefore\ Pnr = A - P.$$

$$\therefore\ r = \frac{A - P}{Pn} = \frac{610 - 500}{2000} = 0.055.$$

$$= 5\tfrac{1}{2}\,\%.$$

366. Since P will in n years amount to A, it is evident that P at the present time may be considered equivalent in value to A due at the end of n years; so that P may be regarded as the *present worth* of a given future sum A.

Find the present worth of \$600, due in 2 years, the rate of interest being 6 per cent.

$$A = P(1 + nr).$$

$$\therefore\ P = \frac{A}{1 + nr} = \frac{\$600}{1 + 0.12} = \$535.71.$$

367. Compound Interest.

I. When compound interest is reckoned payable *annually*,

The amount of P dollars in

$$\begin{aligned} &1 \text{ year is} && P(1+r) \text{ or } PR, \\ &2 \text{ years is} && PR(1+r) \text{ or } PR^2, \\ &n \text{ years is} && PR^n. \end{aligned}$$

That is, $$A = PR^n.$$

Hence, also, $$P = \frac{A}{R^n}.$$

II. When compound interest is payable *semi-annually*,

The amount of P dollars in

$$\tfrac{1}{2} \text{ year is } P\left(1 + \frac{r}{2}\right),$$

$$1 \text{ year is } P\left(1 + \frac{r}{2}\right)^2,$$

$$n \text{ years is } P\left(1 + \frac{r}{2}\right)^{2n}.$$

That is, $$A = P\left(1 + \frac{r}{2}\right)^{2n}.$$

III. When the interest is payable *quarterly*,

$$A = P\left(1 + \frac{r}{4}\right)^{4n}.$$

IV. When the interest is payable *monthly*,

$$A = P\left(1 + \frac{r}{12}\right)^{12n}.$$

V. When interest is payable q times a year,

$$A = P\left(1 + \frac{r}{q}\right)^{qn}.$$

Find the present worth of \$500, due in 4 years, at 5 per cent compound interest.

$$A = P(1 + r)^4.$$

$$\therefore\ P = \frac{A}{(1 + r)^4} = \frac{\$500}{(1.05)^4} = \$411.36.$$

368. Sinking Funds. If the sum set apart at the end of each year to be put at compound interest is represented by S, then

The sum at the end of the

$$\begin{aligned} &\text{first year} &&= S, \\ &\text{second year} &&= S + SR, \\ &\text{third year} &&= S + SR + SR^2, \\ &n\text{th year} &&= S + SR + SR^2 + \cdots\cdots + SR^{n-1}. \end{aligned}$$

That is, the amount $A = S + SR + SR^2 + \cdots\cdots + SR^{n-1}.$

$$\therefore\ AR = SR + SR^2 + SR^3 + \cdots\cdots + SR^n.$$

$$\therefore\ AR - A = SR^n - S.$$

$$\therefore\ A = \frac{S(R^n - 1)}{R - 1},$$

or,

$$A = \frac{S(R^n - 1)}{r}.$$

(1) If \$10,000 be set apart annually, and put at 6 per cent compound interest for 10 years, what will be the amount?

$$A = \frac{S(R^n - 1)}{r} = \frac{\$10,000(1.06^{10} - 1)}{0.06}.$$

By logarithms the amount is found to be \$131,740 (*nearly*).

(2) A county owes \$60,000. What sum must be set apart annually, as a sinking fund, to cancel the debt in 10 years, provided money is worth 6 per cent?

$$S = \frac{Ar}{R^n - 1} = \frac{\$60,000 \times 0.06}{1.06^{10} - 1} = \$4555 \ (\textit{nearly}).$$

NOTE. The amount of tax required yearly is \$3600 for the *interest* and \$4555 for the sinking fund; that is, \$8155.

369. Annuities. A sum of money that is payable yearly, or in parts at fixed periods in the year, is called an **annuity**.

To find the amount of an unpaid annuity when the interest, time, and rate per cent are given.

The sum due at the *end* of the

$$\begin{aligned} &\text{first year} &&= S, \\ &\text{second year} &&= S + SR, \\ &\text{third year} &&= S + SR + SR^2, \\ &n\text{th year} &&= S + SR + SR^2 + \cdots\cdots + SR^{n-1}. \end{aligned}$$

That is,
$$A = \frac{S(R^n - 1)}{r}.$$

An annuity of \$1200 was unpaid for 6 years. What was the amount due if interest is reckoned at 6 per cent?

$$A = \frac{S(R^n - 1)}{r} = \frac{\$1200(1.06^6 - 1)}{0.06} = \$8370.$$

370. *To find the present worth of an annuity when the time it is to continue and the rate per cent are given.*

Let P denote the present worth. Then the amount of P for n years will be equal to A, the amount of the annuity for n years.

Therefore for n years

$$A = P(1 + r)^n = PR^n, \qquad \S\ 367$$

and
$$A = \frac{S(R^n - 1)}{R - 1}. \qquad \S\ 369$$

$$\therefore\ PR^n = \frac{S(R^n - 1)}{R - 1}.$$

$$\therefore\ P = \frac{S}{R^n} \times \frac{R^n - 1}{R - 1}.$$

This equation may be written

$$P = \frac{S}{R-1} \times \frac{R^n - 1}{R^n} = \frac{S}{R-1}\left(1 - \frac{1}{R^n}\right).$$

As n increases, the expression

$$\left(1 - \frac{1}{R^n}\right)$$

approaches 1. Therefore, if the annuity is *perpetual*,

$$P = \frac{S}{R-1} = \frac{S}{r}.$$

(1) Find the present worth of an annual pension of \$105 for 5 years, at 4 per cent interest.

$$\begin{aligned} P &= \frac{S}{R^n} \times \frac{R^n - 1}{R - 1} \\ &= \frac{\$105}{1.04^5} \times \frac{1.04^5 - 1}{1.04 - 1} = \$467\ (\textit{nearly}). \end{aligned}$$

(2) Find the present worth of a perpetual scholarship that pays \$300 annually, at 6 per cent interest.

$$P = \frac{S}{r} = \frac{\$300}{0.06} = \$5000.$$

371. *To find the present worth of an annuity that begins in a given number of years, when the time it is to continue and the rate per cent are given.*

Let p denote the number of years before the annuity begins, and q the number of years the annuity is to continue.

Then the present worth of the annuity to the time it *terminates* is

$$\frac{S}{R^{p+q}} \times \frac{R^{p+q} - 1}{R - 1},$$

and the present worth of the annuity to the time it *begins* is

$$\frac{S}{R^p}\times\frac{R^p-1}{R-1}.$$

Hence,

$$P=\left(\frac{S}{R^{p+q}}\times\frac{R^{p+q}-1}{R-1}\right)-\left(\frac{S}{R^p}\times\frac{R^p-1}{R-1}\right).$$

$$\therefore\ P=\frac{S}{R^{p+q}}\times\frac{R^q-1}{R-1}.$$

If the annuity is to begin at the end of p years, and to be perpetual, the formula

$$P=\frac{S}{R^{p+q}}\times\frac{R^q-1}{R-1}$$

becomes

$$P=\frac{S}{R^p(R-1)}\times\frac{R^q-1}{R^q}.$$

And since $\dfrac{R^q-1}{R^q}$ approaches 1 (§ 370),

$$P=\frac{S}{R^p(R-1)}.$$

(1) Find the present worth of an annuity of \$5000, to begin in 6 years, and to continue 12 years, at 6 per cent interest.

$$P=\frac{S}{R^{p+q}}\times\frac{R^q-1}{R-1}$$

$$=\frac{\$5000}{1.06^{18}}\times\frac{1.06^{12}-1}{0.06}=\$29{,}550.$$

(2) Find the present worth of a perpetual annuity of \$1000, to begin in 3 years, at 4 per cent interest.

$$P=\frac{S}{R^p(R-1)}=\frac{\$1000}{1.04^3\times 0.04}=\$22{,}225.$$

372. *To find the annuity when the present worth, the time, and the rate per cent are given.*

$$P = \frac{S(R^n - 1)}{R^n(R - 1)}. \qquad \S\ 370$$

$$\therefore\ S = \frac{PR^n(R-1)}{R^n - 1} = Pr \times \frac{R^n}{R^n - 1}.$$

What annuity for 5 years will \$4675 give when interest is reckoned at 4 per cent?

$$S = Pr \times \frac{R^n}{R^n - 1} = \$4675 \times 0.04 \times \frac{1.04^5}{1.04^5 - 1} = \$1050.$$

373. Life Insurance. In order that a certain sum may be secured, to be payable at the death of a person, he pays yearly a fixed *premium.*

If P denote the premium to be paid for n years to insure an amount A, to be paid immediately after the last premium, then

$$A = \frac{P(R^n - 1)}{R - 1}. \qquad \S\ 368$$

$$\therefore\ P = \frac{A(R-1)}{R^n - 1} = \frac{Ar}{R^n - 1}.$$

If A is to be paid a year after the last premium, then

$$P = \frac{A(R-1)}{R(R^n - 1)} = \frac{Ar}{R(R^n - 1)}.$$

NOTE. In the calculation of life insurances it is necessary to employ tables which show for any age the probable duration of life.

374. Bonds. If P denote the price of a bond that has n years to run, and bears r per cent interest, S the face of the bond, and q the current rate of interest, what interest on his investment will a purchaser of such a bond receive?

Let x denote the rate of interest on the investment.

Then $P(1 + x)^n$ is the value of the purchase money at the end of n years.

$Sr(1+q)^{n-1}+Sr(1+q)^{n-2}+\cdots\cdots+Sr+S$ is the amount of money received on the bond if the interest received from the bond is put immediately at compound interest at q per cent.

But $\quad Sr(1+q)^{n-1}+Sr(1+q)^{n-2}+\cdots\cdots+Sr+S$

$$=S+\frac{Sr[(1+q)^n-1]}{q}.$$

$$\therefore\ P(1+x)^n=S+\frac{Sr[(1+q)^n-1]}{q}.$$

$$\therefore\ 1+x=\left(\frac{S}{P}+\frac{Sr[(1+q)^n-1]}{Pq}\right)^{\frac{1}{n}}$$

$$\left(\frac{Sq+Sr(1+q)^n-Sr}{Pq}\right)^{\frac{1}{n}}.$$

(1) What interest will a person receive on his investment if he buys at 114 a 4 per cent bond that has 26 years to run, money being worth $3\frac{1}{2}$ per cent?

$$1+x=\left(\frac{3.5+4(1.035)^{26}-4}{3.99}\right)^{\frac{1}{26}}.$$

By logarithms, $1+x=1.033$.

That is, the purchaser will receive $3\frac{1}{3}$ per cent for his money.

(2) At what price must 7 per cent bonds, running 12 years, with the interest payable semi-annually, be bought, in order that the purchaser may receive on his investment 5 per cent, interest semi-annually, which is the current rate of interest?

$$P(1+x)^n=\frac{Sq+Sr(1+q)^n-Sr}{q}.$$

$$\therefore\ P=\frac{Sq+Sr(1+q)^n-Sr}{q(1+x)^n}.$$

In this case $S=100$; and, as the interest is semi-annual,

$$q=0.025,\ r=0.035,\ n=24,\ x=0.025.$$

Hence, $\quad P=\frac{2.5+3.5(1.025)^{24}-3.5}{0.025(1.025)^{24}}.$

By logarithms, $\quad P=118.$

Exercise 120.

1. In how many years will \$100 amount to \$1050, at 5 per cent compound interest?

2. In how many years will \$$A$ amount to \$$B$ (1) at simple interest, (2) at compound interest, r and R being used in their usual sense?

3. Find the difference (to five places of decimals) between the amount of \$1 in 2 years, at 6 per cent compound interest, according as the interest is due yearly or monthly.

4. At 5 per cent, find the amount of an annuity of \$$A$ which has been left unpaid for 4 years.

5. Find the present value of an annuity of \$100 for 5 years, reckoning interest at 4 per cent.

6. A perpetual annuity of \$1000 is to be purchased, to begin at the end of 10 years. If interest is reckoned at $3\frac{1}{2}$ per cent, what should be paid for it?

7. A debt of \$1850 is discharged by two payments of \$1000 each, at the end of one and two years. Find the rate of interest paid.

8. Reckoning interest at 4 per cent, what annual premium should be paid for 30 years, in order to secure \$2000 to be paid at the end of that time, the premium being due at the beginning of each year?

9. An annual premium of \$150 is paid to a life-insurance company for insuring \$5000. If money is worth 4 per cent, for how many years must the premium be paid in order that the company may sustain no loss?

10. What may be paid for bonds due in 10 years, and bearing semi-annual coupons of 4 per cent each, in order to realize 3 per cent semi-annually, if money is worth 3 per cent semi-annually?

11. When money is worth 2 per cent semi-annually, if bonds having 12 years to run, and bearing semi-annual coupons of $3\frac{1}{2}$ per cent each, are bought at $114\frac{1}{8}$, what per cent is realized on the investment?

12. If \$126 is paid for bonds due in 12 years, and yielding $3\frac{1}{2}$ per cent semi-annually, what per cent is realized on the investment, provided money is worth 2 per cent semi-annually?

13. A person borrows \$600.25. How much must he pay annually that the whole debt may be discharged in 35 years, allowing simple interest at 4 per cent?

14. A perpetual annuity of \$100 a year is sold for \$2500. At what rate is the interest reckoned?

15. A perpetual annuity of \$320, to begin 10 years hence, is to be purchased. If interest is reckoned at $3\frac{1}{5}$ per cent, what should be paid for it?

16. A sum of \$10,000 is loaned at 4 per cent. At the end of the first year a payment of \$400 is made; and at the end of each following year a payment is made greater by 30 per cent than the preceding payment. Find in how many years the debt will be paid.

17. A man with a capital of \$100,000 spends every year \$9000. If the current rate of interest is 5 per cent, in how many years will he be ruined?

18. Find the amount of \$365 at compound interest for 20 years, at 5 per cent.

CHAPTER XXVIII.

CHOICE.

375. Fundamental Principle. *If one thing can be done in* **a** *different ways, and, when it has been done, a second thing can be done in* **b** *different ways, then the two things can be done together in* **a** $\times$ **b** *different ways.*

For, corresponding to the *first* way of doing the first thing, there are b different ways of doing the second thing; corresponding to the *second* way of doing the first thing, there are b different ways of doing the second thing; and so on for *each* of the a different ways of doing the first thing. Therefore there are $a \times b$ different ways of doing the two things together.

(1) If a box contains four capital letters, A, B, C, D, and three small letters, x, y, z, in how many different ways may two letters, one a capital letter and one a small letter, be selected?

A capital letter may be selected in four different ways, since any one of the letters A, B, C, D, may be selected. A small letter may be selected in three different ways, since any one of the letters x, y, z, may be selected. Any small letter may be put with any capital letter.

Thus, with A we may put x, or y, or z;
with B we may put x, or y, or z;
with C we may put x, or y, or z;
with D we may put x, or y, or z.

Hence the number of ways in which a selection may be made is 4×3, or 12. These ways are:

Ax	Bx	Cx	Dx
Ay	By	Cy	Dy
Az	Bz	Cz	Dz

(2) On a shelf are 7 English, 5 French, and 9 German books. In how many ways may two books, not in the same language, be selected?

An English book and a French book can be selected in 7×5, or 35, ways. A French book and a German book in 5×9, or 45, ways. An English book and a German book in 7×9, or 63, ways.

Hence, there is a choice of $35 + 45 + 63$, or 143, ways.

(3) Out of the ten figures, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, how many numbers, each consisting of two figures, can be formed?

Since 0 has no value in the left-hand place, the left-hand place can be filled in 9 ways.

The right-hand place can be filled in 10 ways, since *repetitions* of the digits are allowed (as 22, 33, etc.).

Hence, the whole number is 9×10, or 90.

376. By successive application of the principle of § 375 it may be shown that,

If one thing can be done in a *different ways, then a second thing can be done in* b *different ways, then a third thing in* c *different ways, then a fourth thing in* d *different ways, etc., the number of different ways of doing all the things together will be* $a \times b \times c \times d$, *etc.*

For, the first and second things can be done together in $a \times b$ different ways (§ 375), and the third thing in c different ways; hence, (§ 375), the first and second things and the third thing can be done together in $(a \times b) \times c$ different ways. Therefore, the first three things can be done in $a \times b \times c$ different ways. And so on for any number of things.

In how many ways can four Christmas presents be given to four boys, one to each boy?

The first present may be given to any one of the boys; hence there are 4 ways of disposing of it.

The second present may be given to any one of the other three boys; hence there are 3 ways of disposing of it.

The third present may be given to either of the other two boys; hence there are 2 ways of disposing of it.

The fourth present must be given to the last boy; hence there is only 1 way of disposing of it.

There are, then, $4 \times 3 \times 2 \times 1$, or 24, ways.

377. Selections and Arrangements.

(1) In how many ways can a vowel and a consonant be chosen out of the alphabet?

Since there are in the alphabet 6 vowels and 20 consonants, a vowel can be chosen in 6 ways and a consonant in 20 ways, and both (§ 375) in 6×20, or 120, ways.

(2) In how many ways can a two-lettered word be made, containing one vowel and one consonant?

The vowel can be chosen in 6 ways and the consonant in 20 ways; and then each combination of a vowel and a consonant can be written in 2 ways; as *ac*, *ca*.

Hence, the whole number of ways is $6 \times 20 \times 2$, or 240.

These two examples show the difference between a *selection* or *combination* of different things, and an *arrangement* or *permutation* of the same things.

Thus, *ac* forms a selection of a vowel and a consonant, and *ac* and *ca* form two different *arrangements* of this selection.

From (1) it is seen that 120 different selections can be made with a vowel and a consonant; and from (2) it is seen that 240 different *arrangements* can be made with the same.

Again, a, b, c is a selection of three letters from the alphabet. This selection admits of 6 different arrangements, as follows:

$$\begin{array}{ccc} abc & bca & cab \\ acb & bac & cba \end{array}$$

A **selection** or **combination** of any number of things is a group of that number of things put together without regard to their order.

An **arrangement** or **permutation** of any number of things is a group of that number of things put together, regard being paid to their order.

378. Arrangements, Things all Different. *The number of different arrangements (or permutations) of* n *different things taken all together is*

$$n(n-1)(n-2)(n-3)\cdots\cdots 3\times 2\times 1.$$

For, the first place can be filled in n ways, then the second place in $n-1$ ways, then the third place in $n-2$ ways, and so on to the last place, which can be filled in only 1 way.

Hence (§ 376) the whole number of arrangements is the continued product of all these numbers,

$$n(n-1)(n-2)(n-3)\cdots\cdots 3\times 2\times 1.$$

For the sake of brevity this product is written $\underline{|n}$, and is read **factorial n.**

Observe that $1\times 2\cdots\cdots(n-1)n=\underline{|n}$.

How many different arrangements of nine letters each can be formed with the letters in *Cambridge?*

There are nine letters. In making any arrangement any one of the letters can be put in the first place. Hence, the first place can be filled in 9 ways.

Then the second place can be filled with any one of the remaining eight letters; that is, in 8 ways.

In like manner, the third place can be filled in 7 ways, the fourth place in 6 ways, and so on; and, lastly, the ninth place in 1 way.

If the nine places are indicated by Roman numerals, the result is (§ 376) as follows:

$$\begin{array}{ccccccccc} \text{I.} & \text{II.} & \text{III.} & \text{IV.} & \text{V.} & \text{VI.} & \text{VII.} & \text{VIII.} & \text{IX.} \\ 9\times & 8\times & 7\times & 6\times & 5\times & 4\times & 3\ \times & 2\ \times & 1 = 362{,}880 \text{ ways.} \end{array}$$

Hence, there are 362,880 different arrangements possible.

379. *The number of different arrangements of* **n** *different things taken* **r** *at a time is*

$$n(n-1)(n-2) \cdots\cdots \text{ to } r \text{ factors,}$$

that is,
$$n(n-1)(n-2) \cdots\cdots [n-(r-1)],$$

or
$$n(n-1)(n-2) \cdots\cdots (n-r+1).$$

For, the first place can be filled in n ways, the second place in $n-1$ ways, the third place in $n-2$ ways, and the rth place in $n-(r-1)$ ways.

Let $P_{n,r}$ represent the number of arrangements of n different things taken r at a time. Then

$$P_{n,r} = n(n-1)(n-2) \cdots\cdots \text{ to } r \text{ factors}$$
$$= n(n-1)(n-2) \cdots\cdots (n-r+1).$$

How many different arrangements of four letters each can be formed from the letters in *Cambridge?*

There are nine letters and four places to be filled.

The first place can be filled in 9 ways. Then the second place can be filled in 8 ways. Then the third place in 7 ways, and the fourth place in 6 ways.

If the places are indicated by I., II., III., IV., the result is (§ 376)

$$\begin{array}{l} \text{I. II. III. IV.} \\ 9 \times 8 \times 7 \times 6 = 3024 \text{ ways.} \end{array}$$

Hence, there are 3024 different arrangements possible.

380. Selections, Things all Different. *The number of different selections (or combinations) of* **n** *different things taken* **r** *at a time is*

$$\frac{n(n-1)(n-2) \cdots\cdots (n-r+1)}{\lfloor r}.$$

To prove this, let $C_{n,r}$ represent the number of different selections (or combinations) of n different things taken r at a time.

Take one selection of r things; from this selection $\lfloor r$ arrangements can be made (§ 378).

Take a second selection; from this selection $\lfloor r$ arrangements can be made. And so on for *each* of the $C_{n,r}$ selections.

Hence, $C_{n,r} \times \lfloor r$ is the number of *arrangements* of n different things taken r at a time; or

$$C_{n,r} \times \lfloor r = P_{n,r}.$$

$$\therefore\ C_{n,r} = \frac{P_{n,r}}{\lfloor r}.$$

$$\therefore\ C_{n,r} = \frac{n(n-1)(n-2)\cdots\cdots(n-r+1)}{\lfloor r}.$$

Ex. In how many different ways can three vowels be selected from the five vowels a, e, i, o, u?

The number of different ways in which we can *arrange* 3 vowels out of 5 is (§ 376) $5 \times 4 \times 3$, or 60.

These 60 arrangements might be obtained by first forming all the possible selections of 3 vowels out of 5, and then arranging the 3 vowels in each selection in as many ways as possible.

Since each selection can be arranged in $\lfloor 3$, or 6 ways (§ 378), the number of selections is $\frac{60}{6}$ or 10.

The formula applied to this problem gives

$$C_{5,3} = \frac{5 \times 4 \times 3}{1 \times 2 \times 3} = 10.$$

381. Selections, Second Formula. Multiplying both numerator and denominator of the expression for the number of selections in the last example by 2×1, we have

$$C_{5,3} = \frac{5 \times 4 \times 3 \times 2 \times 1}{1 \times 2 \times 3 \times 2 \times 1} = \frac{\lfloor 5}{\lfloor 3 \lfloor 2}.$$

In general, multiplying both numerator and denominator of the expression for $C_{n,r}$ in § 380 by $\lfloor n-r$, we have

$$C_{n,r} = \frac{n(n-1)\cdots\cdots(n-r+1)(n-r)\cdots\cdots 1}{\lfloor r \times (n-r)\cdots\cdots 1}$$

$$= \frac{\lfloor n}{\lfloor r \lfloor n-r}.$$

This second form is more compact than the first, and is more easily remembered.

NOTE. In reducing a result expressed in the above form, it is to be observed that $\lfloor n-r$ cancels all the factors of the numerator from 1 up to and including $n-r$. Thus, in $\frac{\lfloor 12}{\lfloor 5 \lfloor 7}$, $\lfloor 7$ cancels **all** the factors of $\lfloor 12$ from 1 up to and including 7; so that

$$\frac{\lfloor 12}{\lfloor 5 \lfloor 7} = \frac{12 \times 11 \times 10 \times 9 \times 8}{1 \times 2 \times 3 \times 4 \times 5} = 792.$$

382. Theorem. *The number of selections of* **n** *things taken* **r** *at a time is the same as the number of selections of* **n** *things taken* **n − r** *at a time.*

For, $C_{n,n-r} = \dfrac{\lfloor n}{\lfloor n-r \lfloor n-(n-r)} = \dfrac{\lfloor n}{\lfloor n-r \lfloor r} = C_{n,r}.$

This is also evident from the fact that for every selection of r things *taken*, a selection of $n-r$ things is *left*.

Thus, out of 8 things, 3 things can be selected in the same number of ways as 5 things; namely,

$$\frac{\lfloor 8}{\lfloor 3 \lfloor 5} = \frac{8 \times 7 \times 6}{\lfloor 3} = 56 \text{ ways.}$$

NOTE. Evidently $C_{1,1} = 1$; also $C_{1,1} = \dfrac{\lfloor 1}{\lfloor 1 \lfloor 0} = \dfrac{1}{\lfloor 0}$.

$$\therefore \frac{1}{\lfloor 0} = 1, \text{ and } \lfloor 0 = 1.$$

383. Examples in Selections and Arrangements. Of the arrangements possible with the letters of the word *Cambridge*, taken all together,

(1) How many will begin with a vowel?

In filling the nine places of any arrangement the first place can be filled in only 3 ways, the other places in $\lfloor 8$ ways.

Hence, the answer is (§ 376)

$$3 \times \lfloor 8 = 120{,}960.$$

(2) How many will both begin and end with a vowel?

The first place can be filled in 3 ways, the last place in 2 ways (one vowel having been used), and the remaining seven places in $\lfloor 7$ ways.

Hence, the answer is (§ 376)

$$3 \times 2 \times \lfloor 7 = 30{,}240.$$

(3) How many will begin with *Cam?*

The answer is evidently $\lfloor 6$; since our only choice lies in arranging the remaining six letters of the word.

(4) How many will have the letters *c a m* standing together?

This may be resolved into arranging the group *c a m* and the last six letters, regarded as seven distinct elements, and then arranging the letters *c a m*.

The first can be done in $\lfloor 7$ ways, and the second in $\lfloor 3$ ways. Hence both can be done in $\lfloor 7 \times \lfloor 3 = 30{,}240$ ways.

In how many ways can the letters of the word *Cambridge* be written,

(5) Without changing the *place* of any vowel?

The second, sixth, and ninth places can be filled each in only 1 way; the other places in $\lfloor 6$ ways.

Therefore, the whole number of ways is $\lfloor 6 = 720$.

(6) Without changing the *order* of the three vowels?

The vowels in the different arrangements are to be kept in the order, *a*, *i*, *e*.

One of the six consonants can be placed in 4 ways: *before a*, *between a* and *i*, *between i* and *e*, and *after e*.

Then a second consonant can be placed in 5 ways, a third consonant in 6 ways, a fourth consonant in 7 ways, a fifth consonant in 8 ways, and the last consonant in 9 ways. Hence the whole number of ways is

$$4 \times 5 \times 6 \times 7 \times 8 \times 9, \text{ or } 60{,}480.$$

(7) Out of 20 consonants, in how many ways can 18 be selected?

The 18 can be selected in the same number of ways as 2; and the number of ways in which 2 can be selected is

$$\frac{20 \times 19}{2} = 190.$$

(8) In how many ways can the same choice be made so as always to include the letter b?

Taking b first, we must then select 17 out of the remaining 19 consonants. This can be done in

$$\frac{19 \times 18}{2} = 171 \text{ ways.}$$

(9) In how many ways can the same choice be made so as to include b and not to include c?

Taking b first, we have then to choose 17 out of 18, c being excluded. This can be done in 18 ways.

(10) From 20 Republicans and 6 Democrats, in how many ways can 5 different offices be filled, 3 of which must be filled by Republicans, and the other 2 by Democrats?

The first three offices can be assigned to 3 Republicans in

$$20 \times 19 \times 18 = 6840 \text{ ways};$$

and the other two offices can be assigned to 2 Democrats in

$$6 \times 5 = 30 \text{ ways.}$$

There is, then, a choice of $6840 \times 30 = 205{,}200$ different ways.

(11) Out of 20 consonants and 6 vowels, in how many ways can we make a word consisting of 3 different consonants and 2 different vowels?

Three consonants can be selected in $\frac{20 \times 19 \times 18}{1 \times 2 \times 3} = 1140$ ways, and two vowels in $\frac{6 \times 5}{1 \times 2} = 15$ ways. Hence the 5 letters can be selected in $1140 \times 15 = 17{,}100$ ways.

When five letters have been so selected, they can be arranged in $\underline{|5} = 120$ different orders. Hence, there are $17{,}100 \times 120 = 2{,}052{,}000$ different ways of making the word.

Observe that the letters are first *selected* and then *arranged.*

(12) A society consists of 50 members, 10 of whom are physicians. In how many ways can a committee of 6 members be selected so as to include *at least* one physician?

Six members can be selected from the whole society in

$$\frac{\underline{|50}}{\underline{|6}\,\underline{|44}} \text{ ways.}$$

Six members can be selected from the whole society, so as to include *no physician*, by choosing them all from the 40 members who are not physicians, and this can be done in

$$\frac{\underline{|40}}{\underline{|6}\,\underline{|34}} \text{ ways.}$$

Hence, $\dfrac{\underline{|50}}{\underline{|6}\,\underline{|44}} - \dfrac{\underline{|40}}{\underline{|6}\,\underline{|34}}$ is the number of ways of selecting the committee so as to include at least one physician.

384. Greatest Number of Selections. To find for what value of r the number of selections of n things, taken r at a time, is the greatest.

The formula

$$C_{n,r} = \frac{n(n-1)(n-2)\cdots\cdots(n-r+1)}{1 \times 2 \times 3 \times \cdots\cdots r}$$

may be written

$$C_{n,r} = \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \cdots\cdots \frac{n-r+1}{r}.$$

The numerators of the factors on the right side of this equation begin with n, and form a descending series with the common difference 1; and the denominators begin with 1, and form an ascending series with the common difference 1. Therefore, from some point in the series, these factors

become less than 1. Hence, the maximum product is reached when that product includes *all* the factors *greater* than 1.

I. When n is an *odd* number, the numerator and the denominator of each factor will be alternately both odd and both even; so that the factor greater than 1, but nearest to 1, will be the factor whose numerator exceeds the denominator by 2. Hence, in this case, r must have such a value that

$$n-r+1=r+2, \text{ or } r=\frac{n-1}{2}.$$

II. When n is an *even* number, the numerator of the first factor will be even and the denominator odd; the numerator of the second factor will be odd and the denominator even; and so on, alternately; so that the factor greater than 1, but nearest to 1, will be the factor whose numerator exceeds the denominator by 1. Hence, in this case, r must have such a value that

$$n-r+1=r+1, \text{ or } r=\frac{n}{2}.$$

(1) What value of r will give the greatest number of selections out of 7 things?

Here n is odd, and $r=\frac{n-1}{2}=\frac{7-1}{2}=3.$

$$\therefore\ s=\frac{7\times6\times5}{1\times2\times3}=35.$$

If $r=4$, then $$s=\frac{7\times6\times5\times4}{1\times2\times3\times4}=35.$$

When the number of things is *odd*, there will be two equal numbers of selections; namely, when the number of things taken together is *just under* and *just over one-half* of the whole number of things.

(2) What value of r will give the greatest number of selections out of 8 things?

Here n is even, and $r = \frac{n}{2} = \frac{8}{2} = 4.$

$$\therefore\ s = \frac{8 \times 7 \times 6 \times 5}{1 \times 2 \times 3 \times 4} = 70.$$

So that, when the number of things is *even*, the number of selections will be greatest when *one-half* of the whole are taken together.

385. Division into Groups. The number of different ways in which $p + q$ things all different can be divided into *two* groups of p things and q things, respectively, is the same as the number of ways in which p things can be *selected* from $p + q$ things, or $\dfrac{\underline{\lvert p + q}}{\underline{\lvert p}\,\underline{\lvert q}}.$

For, to each selection of p things *taken* corresponds a selection of q things *left*, and each selection therefore effects the division into the required groups.

(1) In how many ways can 18 men be divided into 2 groups of 6 and 12 each?

$$\frac{\underline{\lvert 18}}{\underline{\lvert 6}\,\underline{\lvert 12}}.$$

(2) A boat's crew consists of 8 men, of whom 2 can row only on the stroke side of the boat, and 3 can row only on the bow side. In how many ways can the crew be arranged?

There are left 3 men who can row on either side; 2 of these must row on the stroke side, and 1 on the bow side.

The number of ways in which these three can be divided is

$$\frac{\underline{\lvert 3}}{\underline{\lvert 2}\,\underline{\lvert 1}} = 3 \text{ ways.}$$

When the stroke side is completed, the 4 men can be arranged in $\lfloor\underline{4}$ ways; likewise, the 4 men of the bow side can be arranged in $\lfloor\underline{4}$ ways. Hence the arrangement can be made in

$$3 \times \lfloor\underline{4} \times \lfloor\underline{4} = 1728 \text{ ways.}$$

386. The number of different ways in which $p+q+r$ things all different can be divided into *three* groups of p things, q things, and r things, respectively, is $\dfrac{\lfloor\underline{p+q+r}}{\lfloor\underline{p}\lfloor\underline{q}\lfloor\underline{r}}$.

For, $p+q+r$ things may be divided into two groups of p things and $q+r$ things in $\dfrac{\lfloor\underline{p+q+r}}{\lfloor\underline{p}\lfloor\underline{q+r}}$ ways; then, the group of $q+r$ things may be divided into two groups of q things and r things in $\dfrac{\lfloor\underline{q+r}}{\lfloor\underline{q}\lfloor\underline{r}}$ ways; hence the division into three groups may be effected in

$$\frac{\lfloor\underline{p+q+r}}{\lfloor\underline{p}\lfloor\underline{q+r}} \times \frac{\lfloor\underline{q+r}}{\lfloor\underline{q}\lfloor\underline{r}} \text{ or } \frac{\lfloor\underline{p+q+r}}{\lfloor\underline{p}\lfloor\underline{q}\lfloor\underline{r}} \text{ ways.}$$

And so on for any number of groups.

In how many ways can a company of 100 soldiers be divided into three squads of 50, 30, and 20, respectively?

The answer is $\dfrac{\lfloor\underline{100}}{\lfloor\underline{50}\lfloor\underline{30}\lfloor\underline{20}}$ ways.

387. When the number of things is the *same* in two or more groups, *and there is no distinction to be made between these groups*, the number of ways given by the preceding section is too large.

Divide the letters a, b, c, d, into two groups of two letters each.

The number of ways given by § 386 is $\dfrac{\lfloor\underline{4}}{\lfloor\underline{2}\lfloor\underline{2}} = 6$; these ways are:

I. *ab cd.*	IV. *bc ad.*
II. *ac bd.*	V. *bd ac.*
III. *ad bc.*	VI. *cd ab.*

Since there is no distinction between the groups, group IV. is the same as group III., group V. the same as group II., and group VI. the same as group I. Hence, the correct answer is $\frac{1}{2} \times \frac{\underline{|4}}{\underline{|2}\,\underline{|2}}$, or 3.

In the case of three similar groups the result given by § 386 is to be divided by $\underline{|3}$, the number of ways in which three groups can be arranged among themselves; in the case of four groups, by $\underline{|4}$; and so on.

In how many ways can 18 men be divided into 2 groups of 9 each?

According to § 386, the answer would be $\frac{\underline{|18}}{\underline{|9}\,\underline{|9}}$.

The two groups, considered as groups, have no distinction; therefore, permuting them gives no new arrangement, and the true result is obtained by dividing the preceding by $\underline{|2}$, and is $\frac{\underline{|18}}{\underline{|2}\,\underline{|9}\,\underline{|9}}$.

If any condition be added that will make the two groups *different*, if, for example, one group wear red badges and the other blue, then the answer will be $\frac{\underline{|18}}{\underline{|9}\,\underline{|9}}$.

388. Arrangements, Repetitions allowed. Suppose we have n different letters, and that *repetitions* are allowed.

Then, in making any arrangement, the first place can be filled in n ways; and the second place can be filled in n ways, since repetitions are allowed. Hence the first two places can be filled in $n \times n$, or n^2, ways (§ 375).

Similarly, the first three places can be filled in $n \times n \times n$, or n^3, ways (§ 376).

In general, r places can be filled in n^r ways; or, *the number of arrangements of* n *different things taken* r *at a time, when repetitions are allowed, is* $\mathbf{n}^{\mathbf{r}}$.

(1) How many three-lettered words can be made from the alphabet, when repetitions are allowed?

Here the first place can be filled in 26 ways; the second place in 26 ways; and the third place in 26 ways. The number of words is, therefore, $26^3 = 17,576$.

(2) In the common system of notation, how many numbers can be formed, each number consisting of not more than 5 figures?

Each of the possible numbers may be regarded as consisting of 5 figures, by prefixing zeros to the numbers consisting of less than 5 figures. Thus, 247 may be written 00247.

Hence, every possible arrangement of 5 figures out of the 10 figures, except 00000, will give one of the required numbers; and the answer is $10^5 - 1 = 99,999$; that is, all the numbers between 0 and 100,000.

389. Arrangements, Things Alike, All together. Consider the number of arrangements of the letters a, a, b, b, b, c, d.

Suppose the a's to be different and the b's to be different, and distinguish between them by a_1, a_2, b_1, b_2, b_3.

The seven letters can now be arranged in $\lfloor 7$ ways (§ 376).

Now suppose the two a's to become alike, and the three b's to become alike. Then, where we before had $\lfloor 2$ arrangements of the a's among themselves, we now have but one arrangement, aa; and where we before had $\lfloor 3$ arrangements of the b's among themselves, we now have but one arrangement, bbb.

Hence, the number of arrangements is $\dfrac{\lfloor 7}{\lfloor 2 \lfloor 3} = 420.$

In general, the number of arrangements of n *things, of which* p *are alike,* q *others are alike, and* r *others are alike, ..., is*

$$\frac{\lfloor n}{\lfloor p \lfloor q \lfloor r \cdots\cdots}.$$

(1) In how many ways can the letters of the word *College* be arranged?

If the two l's were different and the two e's were different, the number of ways would be $\lfloor\underline{7}$. Instead of two arrangements of the two l's, we have but one arrangement, ll; and instead of two arrangements of the two e's, we have but one arrangement, ee. Hence, the number of ways is $\dfrac{\lfloor\underline{7}}{\lfloor\underline{2}\lfloor\underline{2}} = 1260.$

(2) In how many different orders can a row of 4 white balls and 3 black balls be arranged?

$$\frac{\lfloor\underline{7}}{\lfloor\underline{4}\lfloor\underline{3}} = 35.$$

390. Selections, Repetitions allowed. We will illustrate by an example the method of solving problems that come under this head.

In how many ways can a selection of 3 letters be made from the letters a, b, c, d, e, if repetitions are allowed?

The selections will be of three classes:

(*a*) All three letters alike.
(*b*) Two letters alike.
(*c*) The three letters all different.

(*a*) There will be 5 selections, since any one of the five letters may be taken three times.

(*b*) Any one of the five letters may be taken twice, and with these may be put any one of the other four letters. Hence, the number of selections is 5×4, or 20.

(*c*) The number of selections (§ 380) is $\dfrac{5 \times 4 \times 3}{1 \times 2 \times 3}$, or 10. Hence, the total number of selections is $5 + 20 + 10 = 35$.

391. Selections and Arrangements, Things Alike. We will illustrate by an example the method of solving problems that come under this head.

How many selections of four letters each can be made from the letters in *Proportion?* How many arrangements of four letters each?

There are 10 letters as follows:

$$\begin{array}{cccccc} o & p & r & t & i & n \\ o & p & r & & & \\ o & & & & & \end{array}$$

Selections:

The selections may be divided into four classes:

(*a*) Three letters alike.
(*b*) Two letters alike, two others alike.
(*c*) Two letters alike, other two different.
(*d*) Four letters different.

(*a*) With the three o's we may put any one of the five other letters, giving 5 selections.

(*b*) We may choose any two out of the three pairs, o, o; p, p; r, r.

$$\frac{3\times 2}{1\times 2}=3 \text{ selections.}$$

(*c*) With any one of the three pairs we can put any two of the five remaining letters in the first line.

$$3\times\frac{5\times 4}{1\times 2}=30 \text{ selections.}$$

(*d*)
$$\frac{6\times 5\times 4\times 3}{1\times 2\times 3\times 4}=15 \text{ selections.}$$

Hence, the total number of *selections* is

$$5+3+30+15=53.$$

Arrangements:

(*a*) Each selection can be arranged in $\dfrac{\lfloor 4}{\lfloor 3}=4$ ways.

$$5\times 4=20 \text{ arrangements.}$$

(*b*) Each selection can be arranged in $\dfrac{\lfloor 4}{\lfloor 2\lfloor 2}=6$ ways.

$$3\times 6=18 \text{ arrangements.}$$

(*c*) Each selection can be arranged in $\dfrac{\lfloor 4}{\lfloor 2}=12$ ways.

$$30\times 12=360 \text{ arrangements.}$$

(*d*) Each selection can be arranged in $\lfloor 4=24$ ways.

$$15\times 24=360 \text{ arrangements.}$$

Hence, the total number of *arrangements* is

$$20+18+360+360=758.$$

392. Total Number of Selections.

I. *The whole number of ways in which a selection (of some, or all) can be made from* n *different things is* $2^n - 1$.

For each thing can be either taken or left; that is, can be disposed of in two ways. There are n things; hence (§ 376) they can all be disposed of in 2^n ways. But among these ways is included the case in which all are rejected; and this case is inadmissible. Hence, the number of ways of making a selection is $2^n - 1$.

How many different amounts can be weighed with 1 lb., 2 lb., 4 lb., 8 lb., and 16 lb. weights?

$$2^5 - 1 = 31.$$

II. *The whole number of ways in which a selection can be made from* $\mathrm{p} + \mathrm{q} + \mathrm{r} \cdots\cdots$ *things, of which* p *are alike,* q *are alike,* r *are alike, etc., is* $(\mathrm{p}+1)(\mathrm{q}+1)(\mathrm{r}+1)\cdots\cdots - 1$.

For the set of p things may be disposed of in $p+1$ ways, since none of them may be taken, or 1, 2, 3, ..., or p, may be taken.

In like manner, the q things may be disposed of in $q+1$ ways; the r things in $r+1$ ways; and so on.

Hence (§ 376) all the things may be disposed of in $(p+1)(q+1)(r+1)\cdots\cdots$ ways.

But the case in which *all* the things are rejected is inadmissible; hence, the whole number of ways is

$$(p+1)(q+1)(r+1)\cdots\cdots - 1.$$

In how many ways can 2 boys divide between them 10 oranges all alike, 15 apples all alike, and 20 peaches all alike?

Here, the case in which the first boy takes none, and the case in which the second boy takes none, must be rejected.

Therefore, the answer is one less than the result, according to II.

$$11 \times 16 \times 21 - 2 = 3694.$$

Exercise 121.

1. How many different permutations can be made of the letters in the word *Ecclesiastical*, taken all together?

2. Of all the numbers that can be formed with four of the digits 5, 6, 7, 8, 9, how many will begin with 56?

3. If the number of permutations of n things, taken 4 together, be equal to 12 times the permutations of n things, taken 2 together, find n.

4. With 3 consonants and 2 vowels, how many words of 3 letters can be formed, beginning and ending with a consonant, and having a vowel for the middle letter?

5. Out of 20 men, in how many different ways can 4 men be chosen to be on guard? In how many of these would one particular man be taken, and from how many would he be left out?

6. Of 12 books of the same size, a shelf will hold 5. How many different arrangements on the shelf may be made?

7. Of 8 men forming a boat's crew, one is selected as stroke. How many arrangements of the rest are possible? When the 4 men who row on each side are decided on, how many arrangements are still possible?

8. How many signals may be made with 6 flags of different colors, which can be hoisted either singly, or any number at a time?

9. How many signals may be made with 8 flags of different colors, which can be hoisted either singly, or any number at a time one above another?

10. How many different signals can be made with 10 flags, of which 3 are white, 2 red, and the rest blue, always hoisted all together and one above another?

11. How many signals can be made with 7 flags, of which 2 are red, 1 white, 3 blue, and 1 yellow, always displayed all together and one above another?

12. In how many different ways may the 8 men serving a field-gun be arranged, so that the same man may always lay the gun?

13. Find the number of signals which can be made with 4 lights of different colors when displayed any number at a time, arranged one above another, side by side, or diagonally.

14. From 10 soldiers and 8 sailors, how many different parties of 3 soldiers and 3 sailors can be formed?

15. How many signals can be made with 3 blue and 2 white flags, which can be displayed either singly, or any number at a time one above another?

16. In how many ways can a party of 6 take their places at a round table?

17. Out of 12 Democrats and 16 Republicans, how many different committees can be formed, each committee consisting of 3 Democrats and 4 Republicans?

18. From 12 soldiers and 8 sailors, how many different parties of 3 soldiers and 2 sailors can be formed?

19. Find the number of combinations of 100 things, 97 together.

20. With 20 consonants and 5 vowels, how many different words can be formed consisting of 3 different consonants and 2 different vowels, any arrangement of letters being considered a word?

21. Of 30 things, how many must be taken together in order that, having that number for selection, there may be the greatest possible variety of choice?

22. There are m things of one kind and n of another; how many different sets can be made containing r things of the first and s of the second?

23. The number of combinations of n things, taken r together, is 3 times the number taken $r-1$ together, and half the number taken $r+1$ together. Find n and r.

24. In how many ways may 12 things be divided into 3 sets of 4?

25. How many words of 6 letters may be formed of 3 vowels and 3 consonants, the vowels always having the even places?

26. From a company of 90 men, 20 are detached for mounting guard each day. How long will it be before the same 20 men are on guard together, supposing the men to be changed as much as possible; and how many times will each man have been on guard?

27. Supposing that a man can place himself in 3 distinct attitudes, how many signals can be made by 4 men placed side by side?

28. How many different arrangements may be made of 11 cricketers, supposing the same 2 always to bowl?

29. Five flags of different colors can be hoisted either singly, or any number at a time one above another. How many different signals can be made with them?

30. How many signals can be made with 5 lights of different colors, which can be displayed either singly, or any number at a time side by side, or one above another?

31. The number of permutations of n things, 3 at a time, is 6 times the number of combinations, 4 at a time. Find n.

CHAPTER XXIX.

CHANCE.

393. Definitions. If an event can happen in a ways and fail in b ways, and all these $a+b$ ways are *equally likely* to occur; if, also, one, *and only one*, of these $a+b$ ways *can* occur, and one *must* occur; then, the **chance** of the event *happening* is $\frac{a}{a+b}$, and the chance of the event *failing* is $\frac{b}{a+b}$. Hence,

I. *The chance of an event happening is expressed by the fraction of which the numerator is the number of favorable ways, and the denominator the whole number of ways favorable and unfavorable.*

Thus, if 1 ball is drawn from a bag containing 3 white balls and 9 black balls, the chance of drawing a white ball is $\frac{3}{12}$; or, as it is expressed, one chance in four.

II. *The chance of an event not happening is expressed by the fraction of which the numerator is the number of unfavorable ways, and the denominator the whole number of ways favorable and unfavorable.*

Thus, if 1 ball is drawn from a bag containing 3 white and 9 black balls, the chance that it is *not* a white ball is $\frac{9}{12}$.

Again, since $$\frac{a}{a+b}+\frac{b}{a+b}=1,$$

we have $$\frac{b}{a+b}=1-\frac{a}{a+b}.$$

Hence, if p is the chance of an event happening, $1-p$ is the chance of the event failing.

394. Certainty. If the event is *certain* to happen, there are no ways of failing, and $1-p=0$, therefore $p=1$. Hence *certainty* is expressed by 1.

395. Odds. If a denote the favorable and b the unfavorable ways of an event, the *odds* are said to be a to b *in favor of* the event, if a is greater than b; and b to a *against* the event, if b is greater than a; and *even* on the event, if a is equal to b.

396. Examples. Simple Events.

(1) What is the chance of throwing double sixes in one throw with two dice?

Each die may fall in 6 ways, and all these ways are equally likely to occur. Hence, the two dice may fall in 6×6, or 36, ways (§ 376), and these 36 ways are all equally likely to occur. Moreover, only *one* of the 36 ways *can* occur, and one *must* occur.

There is only one way which will give double sixes. Hence the chance of throwing double sixes is $\frac{1}{36}$.

(2) What is the chance of throwing one, and only one, five in one throw with two dice?

The whole number of ways, all equally likely to occur, in which the dice can fall is 36. In 5 of these 36 ways the first die will be a five, and the second die not a five; in 5 of these 36 ways the second die will be a five, and the first not a five. Hence, in 10 of these ways one die, and only one die, will be a five; and the required chance is $\frac{10}{36}$, or $\frac{5}{18}$. Hence, the odds are 13 to 5 against the event.

(3) In the same problem, what is the chance of throwing *at least* one five?

Here we have to include also the way in which both dice fall fives, and the required chance is $\frac{11}{36}$.

(4) What is the chance of throwing a total of 5 in one throw with two dice?

The whole number of ways, all equally likely to occur, in which the dice can fall is 36. Of these ways 4 give a total of 5; viz., 1 and 4, 2 and 3, 3 and 2, 4 and 1. Hence, the required chance is $\frac{4}{36}$, or $\frac{1}{9}$.

(5) From an urn containing 5 black and 4 white balls, 3 balls are to be drawn at random. Find the chance that 2 balls will be black and 1 white.

There are 9 balls in the urn. The whole number of ways in which 3 balls can be selected from 9 is $\frac{9 \times 8 \times 7}{1 \times 2 \times 3}$, or 84.

From the 5 black balls 2 can be selected in $\frac{5 \times 4}{1 \times 2}$, or 10 ways; from the 4 white balls 1 can be selected in 4 ways; hence, 2 black balls and 1 white ball can be selected in 10×4, or 40 ways.

The required chance is $\frac{40}{84} = \frac{10}{21}$.

Therefore the odds are 11 to 10 against the event.

(6) From a bag containing 10 balls, 4 are drawn and replaced; then 6 are drawn. Find the chance that the 4 first drawn are among the 6 last drawn.

The second drawing could be made altogether in

$$\frac{\lfloor 10}{\lfloor 6 \lfloor 4} = 210 \text{ ways.}$$

But the drawing can be made so as to include the 4 first drawn in

$$\frac{\lfloor 6}{\lfloor 2 \lfloor 4} = 15 \text{ ways,}$$

since the only choice consists in selecting 2 balls from the 6 not previously drawn. Hence, the required chance is $\frac{15}{210} = \frac{1}{14}$.

(7) If 4 coppers are tossed, what is the chance that exactly 2 will turn up heads?

Since each coin may fall in 2 ways, the 4 coins may fall in $2^4 = 16$ ways (§ 388). The 2 coins to turn up heads can be selected from the 4 coins in $\frac{4 \times 3}{1 \times 2} = 6$ ways. Hence, the required chance is $\frac{6}{16} = \frac{3}{8}$.

Therefore the odds are 5 to 3 against the event.

(8) In one throw with two dice which sum is more likely to be thrown, 9 or 12?

Out of the 36 possible ways of falling, *four* give the sum 9 (namely, $6 + 3$, $3 + 6$, $5 + 4$, $4 + 5$), and *only one* way gives 12 (namely, $6 + 6$). Hence, the chance of throwing 9 is *four times* that of throwing 12.

Exercise 122.

1. The chance of an event happening is $\frac{4}{7}$. What are the odds in favor of the event?

2. If the odds are 10 to 1 against an event, what is the chance of its happening?

3. In one throw with a pair of dice what number is most likely to be thrown? Find the odds against throwing that number.

4. Find the chance of throwing doublets in one throw with a pair of dice.

5. If 10 persons stand in a line, what is the chance that 2 assigned persons will stand together?

6. If 10 persons form a ring, what is the chance that 2 assigned persons will stand together?

7. Three balls are to be drawn from an urn containing 5 black, 3 red, and 2 white balls. What is the chance of drawing 1 red and 2 black balls?

8. In a bag are 5 white and 4 black balls. If 4 balls are drawn out, what is the chance that they will be all of the same color?

9. If 2 tickets are drawn from a package of 20 tickets marked 1, 2, 3, ..., what is the chance that both will be marked with *odd* numbers?

10. From a bag containing 3 white, 4 black, and 5 red balls, 3 balls are drawn. Find the odds against the 3 being of three different colors.

11. There are 10 tickets numbered 1, 2, ... 9, 0. Three tickets are drawn at random. Find the chance of drawing a total of 22.

NOTE. For a more extended discussion of probability, see Wentworth's College Algebra.

CHAPTER XXX.

CONTINUED FRACTIONS.

397. A fraction in the form

$$\cfrac{a}{b+\cfrac{c}{d+\cfrac{e}{f+\text{etc.}}}}$$

is called a **continued fraction**, though the term is commonly restricted to a continued fraction that has 1 for each of its numerators, as

$$\cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{r+\text{etc.}}}}$$

We shall consider in this chapter some of the elementary properties of such fractions.

398. *Any proper fraction in its lowest terms may be converted into a terminated continued fraction.*

Let $\frac{b}{a}$ be such a fraction; then, if p is the quotient and c the remainder of $a \div b$,

$$\frac{b}{a}=\frac{1}{\frac{a}{b}}=\frac{1}{p+\frac{c}{b}};$$

if q is the quotient and d the remainder of $b \div c$,

$$\frac{1}{p+\frac{c}{b}}=\cfrac{1}{p+\cfrac{1}{\frac{b}{c}}}=\cfrac{1}{p+\cfrac{1}{q+\frac{d}{c}}}.$$

Hence,
$$\frac{b}{a}=\cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{r+\text{etc.}}}}$$

The successive steps of the process are the same as the steps for finding the H. C. F. of a and b; and since a and b are prime to each other, a remainder, 1, will at length be reached, and the fraction terminates.

Observe that p, q, r, ..., are all positive integers.

399. Convergents. The fractions formed by taking one, two, three, ..., of the quotients p, q, r, ..., are

$$\frac{1}{p},\quad \cfrac{1}{p+\cfrac{1}{q}},\quad \cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{r}}},\ \ldots\ldots,$$

which simplified are

$$\frac{1}{p},\quad \frac{q}{pq+1},\quad \frac{qr+1}{(pq+1)r+p},\ \ldots\ldots,$$

and are called the first, second, and third convergents, respectively.

400. *The successive convergents are alternately greater than and less than the true value of the given fraction.*

Let x be the true value of

$$\cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{r+\text{etc.}}}};$$

then, since p, q, r, ..., are positive integers,

$$p<p+\cfrac{1}{q+\cfrac{1}{r+\text{etc.}}}$$

$$\therefore \frac{1}{p} > \cfrac{1}{p + \cfrac{1}{q + \cfrac{1}{r + \text{etc.}}}}; \text{ that is, } \frac{1}{p} > x.$$

Again,
$$q < q + \frac{1}{r} + \text{etc.}$$

$$\therefore \frac{1}{q} > \cfrac{1}{q + \cfrac{1}{r + \text{etc.}}}$$

$$\therefore \cfrac{1}{p + \cfrac{1}{q}} < \cfrac{1}{p + \cfrac{1}{q + \cfrac{1}{r + \text{etc.}}}};$$

that is,
$$\cfrac{1}{p + \cfrac{1}{q}} < x; \text{ and so on.}$$

401. *If* $\frac{\mathbf{u}_1}{\mathbf{v}_1}, \frac{\mathbf{u}_2}{\mathbf{v}_2}, \frac{\mathbf{u}_3}{\mathbf{v}_3}$ *are any three consecutive convergents, and if* $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$ *are the quotients that produced them, then* $\frac{\mathbf{u}_3}{\mathbf{v}_3} = \frac{\mathbf{m}_3\mathbf{u}_2 + \mathbf{u}_1}{\mathbf{m}_3\mathbf{v}_2 + \mathbf{v}_1}$.

For, if the first three quotients are p, q, r, the first three convergents are (§ 399),

$$\frac{1}{p}, \quad \frac{q}{pq+1}, \quad \frac{qr+1}{(pq+1)r+p} \ \cdots\cdots \qquad (1)$$

From (§ 399) it is seen that the second convergent is formed from the first by writing in it $p + \frac{1}{q}$ for p; and the third from the second by writing $q + \frac{1}{r}$ for q. In this way, any convergent may be formed from the preceding convergent.

Therefore, $\frac{u_3}{v_3}$ will be formed from $\frac{u_2}{v_2}$ by writing $m_2 + \frac{1}{m_3}$ for m_2.

In (1) it is seen that the third convergent has its numerator $= r \times$ (second numerator) $+$ (first numerator); and its denominator $= r \times$ (second denominator) $+$ (first denominator).

Assume that this law holds true for the third of the three consecutive convergents

$$\frac{u_0}{v_0},\ \frac{u_1}{v_1},\ \frac{u_2}{v_2},\ \text{so that,}\ \frac{u_2}{v_2} = \frac{m_2u_1 + u_0}{m_2v_1 + v_0}. \tag{2}$$

Then, since $\frac{u_3}{v_3}$ is formed from $\frac{u_2}{v_2}$ by using $m_2 + \frac{1}{m_3}$ for m_2,

$$\frac{u_3}{v_3} = \frac{\left(m_2 + \frac{1}{m_3}\right)u_1 + u_0}{\left(m_2 + \frac{1}{m_3}\right)v_1 + v_0} = \frac{m_3(m_2u_1 + u_0) + u_1}{m_3(m_2v_1 + v_0) + v_1}.$$

Substitute u_2 and v_2 for their values $m_2u_1 + u_0$ and $m_2v_1 + v_0$; then

$$\frac{u_3}{v_3} = \frac{m_3u_2 + u_1}{m_3v_2 + v_1}.$$

Therefore the law still holds true; and as it has been shown to be true for the third convergent, the law is general.

402. *The difference between two consecutive convergents* $\frac{u_1}{v_1}$ *and* $\frac{u_2}{v_2}$ *is* $\frac{1}{v_1v_2}$.

The difference between the first two convergents is

$$\frac{1}{p} - \frac{q}{pq+1} = \frac{1}{p(pq+1)}.$$

Let the sign $\sim$ mean *the difference between*, and assume the proposition true for $\frac{u_0}{v_0}$ and $\frac{u_1}{v_1}$;

then $$\frac{u_0}{v_0} \sim \frac{u_1}{v_1} = \frac{u_0 v_1 \sim u_1 v_0}{v_0 v_1} = \frac{1}{v_0 v_1}.$$

But

$$\frac{u_2}{v_2} \sim \frac{u_1}{v_1} = \frac{u_2 v_1 \sim u_1 v_2}{v_1 v_2} = \frac{(m_2 u_1 + u_0) v_1 \sim u_1 (m_2 v_1 + v_0)}{v_1 v_2}$$

(substituting for u_2 and v_2 their values, $m_2 u_1 + u_0$ and $m_2 v_1 + v_0$).

Reducing, $$\frac{u_2}{v_2} \sim \frac{u_1}{v_1} = \frac{u_0 v_1 \sim u_1 v_0}{v_1 v_2},$$

$$= \frac{1}{v_1 v_2} \text{ (by assumption).}$$

Hence, if the proposition be true for one pair of consecutive convergents, it will be true for the next pair; but it has been shown to be true for the *first* pair, therefore it is true for *every* pair.

Since by § 400 the true value of x lies between two consecutive convergents, $\frac{u_1}{v_1}$ and $\frac{u_2}{v_2}$, the convergent $\frac{u_1}{v_1}$ will differ from x by a number less than $\frac{u_1}{v_1} \sim \frac{u_2}{v_2}$; that is, by a number less than $\frac{1}{v_1 v_2}$; so that the error in taking $\frac{u_1}{v_1}$ for x is less than $\frac{1}{v_1 v_2}$, and therefore less than $\frac{1}{v_1^2}$, as $v_2 > v_1$ since $v_2 = m_2 v_1 + v_0$.

Any convergent, $\frac{u_1}{v_1}$, is in its lowest terms; for, if u_1 and v_1 had any common factor, it would also be a factor of $u_1 v_2 \sim u_2 v_1$; that is, a factor of 1.

403. *The successive convergents approach more and more nearly to the true value of the continued fraction.*

Let $\frac{u_0}{v_0}$, $\frac{u_1}{v_1}$, $\frac{u_2}{v_2}$ be consecutive convergents.

Now $\frac{u_2}{v_2}$ differs from x, the true value of the fraction, only because m_2 is used instead of $m_2 + \frac{1}{m_3 + \text{etc.}}$.

Let this complete quotient, which is always greater than unity, be represented by M.

Then, since $\quad \frac{u_2}{v_2} = \frac{m_2 u_1 + u_0}{m_2 v_1 + v_0}, \quad x = \frac{M u_1 + u_0}{M v_1 + v_0}.$

$$\therefore\ x \sim \frac{u_1}{v_1} = \frac{Mu_1 + u_0}{Mv_1 + v_0} \sim \frac{u_1}{v_1} = \frac{u_0 v_1 \sim u_1 v_0}{v_1(Mv_1 + v_0)} = \frac{1}{v_1(Mv_1 + v_0)}.$$

And

$$\frac{u_0}{v_0} \sim x = \frac{u_0}{v_0} \sim \frac{Mu_1 + u_0}{Mv_1 + v_0} = \frac{M(u_0 v_1 \sim u_1 v_0)}{v_0(Mv_1 + v_0)} = \frac{M}{v_0(Mv_1 + v_0)}.$$

Now $1 < M$ and $v_1 > v_0$, and for both these reasons

$$x \sim \frac{u_1}{v_1} < \frac{u_0}{v_0} \sim x.$$

That is, $\frac{u_1}{v_1}$ is nearer to x than $\frac{u_0}{v_0}$ is.

404. *Any convergent* $\frac{\mathbf{u}_1}{\mathbf{v}_1}$ *is nearer the true value* $\mathbf{x}$ *than any other fraction with smaller denominator.*

Let $\frac{a}{b}$ be a fraction in which $b < v_1$.

If $\frac{a}{b}$ is one of the convergents, $x \sim \frac{a}{b} > \frac{u_1}{v_1} \sim x.$ § 403

If $\frac{a}{b}$ is not one of the convergents, and is nearer to x

than $\frac{u_1}{v_1}$ is, then, since x lies between $\frac{u_1}{v_1}$ and $\frac{u_2}{v_2}$ (§ 400), $\frac{a}{b}$ must be nearer to $\frac{u_2}{v_2}$ than $\frac{u_1}{v_1}$ is; that is,

$$\frac{a}{b} \sim \frac{u_2}{v_2} < \frac{u_1}{v_1} \sim \frac{u_2}{v_2}, \text{ or } \frac{v_2 a \sim u_2 b}{v_2 b} < \frac{1}{v_1 v_2};$$

and since $b < v_1$, this would require that $v_2 a \sim u_2 b < 1$; but $v_2 a \sim u_2 b$ cannot be less than 1, for a, b, u_2, v_2 are all integers. Hence, $\frac{u_1}{v_1}$ is nearer to x than $\frac{a}{b}$ is.

405. Find the continued fraction equal to $\frac{31}{75}$, and also the successive convergents.

Following the process of finding the H. C. F. of 31 and 75, the successive quotients are found to be 2, 2, 2, 1, 1, 2. Hence the continued fraction is

$$\cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}}}}$$

To find the successive convergents:

Write the successive quotients in line, $\frac{0}{1}$ under the first quotient, $\frac{1}{2}$ under the second quotient, and then (§ 401) multiply each term by the quotient above it, and add the term to the left to obtain the corresponding term to the right. Thus,

$$\begin{array}{ll} \text{Quotients} & = 2,\ 2,\ 2,\ 1,\ 1,\ 2. \\ \text{Convergents} & = \frac{0}{1},\ \frac{1}{2},\ \frac{2}{5},\ \frac{5}{12},\ \frac{7}{17},\ \frac{12}{29},\ \frac{31}{75}. \end{array}$$

It is convenient to begin to reckon with $\frac{0}{1}$, but the next convergent, in this case $\frac{1}{2}$, is called the *first* convergent.

NOTE. Continued fractions are often written in a more compact form. Thus, the above fraction may be written

$$\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2}$$

406. A quadratic surd may be expressed in the form of a *non-terminating* continued fraction.

To express $\sqrt{3}$ in the form of a continued fraction.

Suppose $\sqrt{3} = 1 + \frac{1}{x}$ (as 1 is the greatest integer in $\sqrt{3}$);

then $\frac{1}{x} = \sqrt{3} - 1.$

$$\therefore\ x = \frac{1}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{2}.$$

Suppose $\frac{\sqrt{3}+1}{2} = 1 + \frac{1}{y}\left(\text{as 1 is the greatest integer in } \frac{\sqrt{3}+1}{2}\right)$;

then $\frac{1}{y} = \frac{\sqrt{3}+1}{2} - 1 = \frac{\sqrt{3}-1}{2}.$

$$\therefore\ y = \frac{2}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{1}.$$

Suppose $\frac{\sqrt{3}+1}{1} = 2 + \frac{1}{z}\left(\text{as 2 is the greatest integer in } \frac{\sqrt{3}+1}{1}\right)$;

then $\frac{1}{z} = \frac{\sqrt{3}+1}{1} - 2 = \sqrt{3} - 1.$

$$\therefore\ z = \frac{1}{\sqrt{3}-1}.$$

This is the same as x above; hence, the quotients 1, 2, will be continually repeated.

$$\therefore\ \sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2} + \text{etc.}}$$

of which $\cfrac{1}{1+\cfrac{1}{2}}$ will be continually repeated, and the whole expression may be written,

$$1 + \frac{\dot{1}}{1+}\,\frac{\dot{1}}{2}.$$

The convergents will be 1, 2, $\frac{5}{3}$, $\frac{7}{4}$, $\frac{19}{11}$, $\frac{26}{15}$, $\frac{71}{41}$, etc.

407. A continued fraction in which the denominators recur is called a **periodic** continued fraction.

The value of a periodic continued fraction can be expressed as the root of a quadratic equation.

Find the surd value of $\dot{\frac{1}{1}}+\dot{\frac{1}{2}}$.

Let x be the value;

then
$$x=\cfrac{1}{1+\cfrac{1}{2+x}}=\frac{2+x}{3+x};$$
$$\therefore\ x^2+2x=2,$$
$$x=-1+\sqrt{3}.$$

We take the + sign since x is evidently positive.

408. An exponential equation can be solved by continued fractions.

Solve by continued fractions $10^x=2$.

Suppose $\qquad x=0+\dfrac{1}{y};$

then $\qquad 10^{\frac{1}{y}}=2,$

or $\qquad 10=2^y.$

$$\therefore\ y=3+\frac{1}{z}\ (\text{as 10 lies between } 2^3 \text{ and } 2^4).$$

Then $\qquad 10=2^{3+\frac{1}{z}}=2^3\times 2^{\frac{1}{z}};$

or $\qquad 2^{\frac{1}{z}}=\frac{10}{8}=\frac{5}{4},$

and $\qquad 2=(\frac{5}{4})^z.$

$$\therefore\ z=3+\frac{1}{u}\left[\text{as 2 lies between }\left(\frac{5}{4}\right)^3\text{ and }\left(\frac{5}{4}\right)^4\right].$$

Then $\qquad 2=(\frac{5}{4})^{3+\frac{1}{u}}=(\frac{5}{4})^3\times(\frac{5}{4})^{\frac{1}{u}};$

or $\qquad (\frac{5}{4})^{\frac{1}{u}}=\frac{128}{125},$

and $\qquad \frac{5}{4}=(\frac{128}{125})^u.$

The greatest integer in u will be found to be 9.

Hence, $$x = 0 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{9 + \text{etc.}}}}$$

The successive convergents will be $\frac{1}{3}$, $\frac{3}{10}$, $\frac{28}{93}$, etc.

The last gives $x = \frac{28}{93} = 0.3010$, *nearly.*

Observe that by the above process we have calculated the common logarithm of 2. By § 402 the error, when 0.3010 is taken for the common logarithm of 2, is considerably less than $\frac{1}{(93)^2}$; that is, considerably less than 0.00011; so that 0.3010 is certainly correct to three places of decimals, and probably correct to four places.

Logarithms are, however, much more easily calculated by the use of series, as will be shown in a following chapter.

Exercise 123.

1. Find continued fractions for $\frac{123}{157}$; $\frac{159}{47}$; $\sqrt{5}$; $\sqrt{11}$; $4\sqrt{6}$; and find the fourth convergent to each.

2. Find continued fractions for $\frac{47}{257}$; $\frac{457}{204}$; $\frac{2065}{4626}$; $\frac{2991}{568}$; and find the third convergent to each.

3. Find continued fractions for $\sqrt{21}$; $\sqrt{22}$; $\sqrt{33}$; $\sqrt{55}$.

4. Obtain convergents, with only two figures in the denominator, that approach nearest to the values of $\sqrt{10}$; $\sqrt{15}$; $\sqrt{17}$; $\sqrt{18}$; $\sqrt{20}$.

5. Find the proper fraction which, if converted into a continued fraction, will have quotients 1, 7, 5, 2.

6. Find the next convergent when the two preceding convergents are $\frac{3}{14}$ and $\frac{19}{89}$, and the next quotient is 5.

7. Find a series of fractions to the ratio of a yard to a meter, if a meter equal 1.0936 yards.

8. If the pound troy is the weight of 22.8157 inches of water, and the pound avoirdupois of 27.7274 inches, find a fraction with denominator < 100 which shall differ from their ratio by < 0.0001.

9. The ratio of the diagonal to a side of a square being $\sqrt{2}$, find a fraction with denominator < 100 which shall differ from their ratio by < 0.0001.

10. The ratio of the circumference of a circle to its diameter being 3.14159265, find the first three convergents, and determine to how many decimal places each may be depended upon as agreeing with the true value.

11. Two scales whose zero points coincide have the distances between consecutive divisions of the one to those of the other as $1 : 1.06577$. Find what division-points most nearly coincide.

12. Find the surd values of

$$3+\frac{\dot{1}}{1+}\,\frac{\dot{1}}{6}; \quad \frac{\dot{1}}{3+}\,\frac{1}{1+}\,\frac{\dot{1}}{6}; \quad 1+\frac{\dot{1}}{2+}\,\frac{1}{3+}\,\frac{\dot{1}}{4}.$$

13. Show that the ratio of the diagonal of a cube to its edge may be nearly expressed by $97 : 56$. Find the limit of the error made in taking this ratio for the true ratio.

14. Find a series of fractions converging to the ratio of 5 hours 48 minutes 51 seconds to 24 hours.

15. Find a series of fractions converging to the ratio of a cubic yard to a cubic meter, if 1 cubic yard $= \frac{76453}{100000}$ of a cubic meter.

CHAPTER XXXI.

SCALES OF NOTATION.

409. Definitions. Let any positive integer be selected as a **radix** or **base**; then any number may be expressed as a polynomial of which the terms are multiples of powers of the radix.

Any positive integer may be selected as the radix; and to each radix corresponds a **scale of notation**.

In writing the polynomials they are arranged by descending powers of the radix, and the powers of the radix are omitted, the *place* of each digit indicating of what power of the radix it is the coefficient.

Thus, in the scale of ten, 2356 stands for

$$2 \times 10^3 + 3 \times 10^2 + 5 \times 10 + 6;$$

in the scale of seven for

$$2 \times 7^3 + 3 \times 7^2 + 5 \times 7 + 6;$$

in the scale of r for

$$2r^3 + 3r^2 + 5r + 6.$$

410. Computation. Computations are made with numbers in any scale, by observing that one unit of any order is equal to the radix-number of units of the next lower order; and that the radix-number of units of any order is equal to one unit of the next higher order.

(1) Add 56,432 and 15,646 (scale of seven).

$$\begin{array}{r} 56432 \\ 15646 \\ \hline 105411 \end{array}$$

The process differs from that in the decimal scale only in that when a sum greater than *seven* is reached, we divide by *seven* (not ten), set down the remainder, and carry the quotient to the next column.

(2) Subtract 34,561 from 61,235 (scale of eight).

$$\begin{array}{r} 61235 \\ 34561 \\ \hline 24454 \end{array}$$

We add eight, instead of ten as in the common scale.

(3) Multiply 5732 by 428 (scale of nine).

$$\begin{array}{r} 5732 \\ 428 \\ \hline 51477 \\ 12564 \\ 25238 \\ \hline 2712127 \end{array}$$

We divide each time by *nine*, set down the remainder, and carry the quotient.

(4) Divide 2,612,127 by 5732 (scale of nine).

$$\begin{array}{l} 5732)\,2612127\,(428 \\ \underline{24238} \\ 17722 \\ \underline{12564} \\ 51477 \\ \underline{51477} \end{array}$$

411. Integers in Any Scale. *If* r *be any positive integer, any positive integer* **N** *may be expressed in the form*

$$N = ar^n + br^{n-1} + \cdots\cdots + pr^2 + qr + s,$$

in which the coefficients a, b, c, ..., *are positive integers, each less than* **r**.

For, divide N by r^n, the highest power of r contained in N, and let the quotient be a with the remainder N_1.

Then, $N = ar^n + N_1$.

In like manner, $N_1 = br^{n-1} + N_2$; $N_2 = cr^{n-2} + N_3$; and so on.

By continuing this process, a remainder s will at length be reached which is less than r. So that,

$$N = ar^n + br^{n-1} + \cdots\cdots + pr^2 + qr + s.$$

Some of the coefficients s, q, p, ... may vanish, and each will be less than r; that is, their values may range from zero to $r-1$. Hence, including zero, r digits will be required to express numbers in the scale of r.

To express any positive integer **N** *in the scale of* **r**.

It is required to express N in the form

$$ar^n + br^{n-1} + \cdots\cdots + pr^2 + qr + s,$$

and to show how the digits a, b, ... may be found.

If $$N = ar^n + br^{n-1} + \cdots\cdots + pr^2 + qr + s,$$

then $$\frac{N}{r} = ar^{n-1} + br^{n-2} + \cdots\cdots + pr + q + \frac{s}{r}.$$

That is, the remainder on dividing N by r is s, the last digit.

Let $$N_1 = ar^{n-1} + br^{n-2} + \cdots\cdots + pr + q;$$

then $$\frac{N_1}{r} = ar^{n-2} + br^{n-3} + \cdots\cdots + p + \frac{q}{r}.$$

That is, the remainder is q, the last but one of the digits.

Hence, to express an integral number in a proposed scale,

Divide the number by the radix, then the quotient by the radix, and so on; the successive remainders will be the successive digits beginning with the units' place.

(1) Express 42,897 (scale of ten) in the scale of six.

$$\begin{array}{r l}
6\underline{)42897} & \\
6\underline{)7149} & \ldots 3 \\
6\underline{)1191} & \ldots 3 \\
6\underline{)198} & \ldots 3 \\
6\underline{)33} & \ldots 0 \\
5 & \ldots 3
\end{array}$$

Ans. 530,333.

(2) Change 37,214 from the scale of eight to the scale of nine.

$$
\begin{array}{r l}
9\underline{)37214} & \\
9\underline{)3363} & \ldots 1 \\
9\underline{)305} & \ldots 6 \\
9\underline{)25} & \ldots 8 \\
2 & \ldots 3 \\
\textit{Ans. } 23{,}861. &
\end{array}
$$

The radix is 8; and hence the two digits on the left, 37, do not mean *thirty-seven*, but $3 \times 8 + 7$, or *thirty-one*, which contains 9 three times, with the remainder 4.

The next partial dividend is $4 \times 8 + 2 = 34$, which contains 9 three times, with the remainder 7; and so on.

(3) In what scale is 140 (scale of ten) expressed by 352?

Let r be the radix; then, in the scale of ten,

$$140 = 3r^2 + 5r + 2 \quad \text{or} \quad 3r^2 + 5r = 138.$$

Solving, we find $r = 6$.

The other value of r is fractional, and therefore inadmissible, since the radix is always a positive integer.

412. Radix-Fractions. As in the decimal scale decimal fractions are used, so in any scale *radix-fractions* are used.

Thus, in the decimal scale, 0.2341 stands for

$$\frac{2}{10} + \frac{3}{10^2} + \frac{4}{10^3} + \frac{1}{10^4};$$

and in the scale of r it stands for

$$\frac{2}{r} + \frac{3}{r^2} + \frac{4}{r^3} + \frac{1}{r^4}.$$

(1) Express $\frac{245}{256}$ (scale of ten) by a radix-fraction in the scale of eight.

Assume $$\frac{245}{256} = \frac{a}{8} + \frac{b}{8^2} + \frac{c}{8^3} + \frac{d}{8^4} + \cdots\cdots$$

Multiply by 8, $$7\tfrac{21}{32} = a + \frac{b}{8} + \frac{c}{8^2} + \frac{d}{8^3} + \cdots\cdots$$

$\therefore a = 7$, and $$\frac{21}{32} = \frac{b}{8} + \frac{c}{8^2} + \frac{d}{8^3} + \cdots\cdots$$

Multiply by 8, $5\frac{1}{4} = b + \frac{c}{8} + \frac{d}{8^2} + \cdots\cdots$

$\therefore\ b = 5$, and $\frac{1}{4} = \frac{c}{8} + \frac{d}{8^2} + \cdots\cdots$

Multiply by 8, $2 = c + \frac{d}{8} + \cdots\cdots$

$\therefore\ c = 2$, and $0 = d$, etc.

The answer is 0.752.

(2) Change 35.14 from the scale of eight to the scale of six.

We take the integral and fractional parts separately.

Integral part:

$$\begin{array}{r|l} 6 & 35 \\ \hline & 4 \quad 5. \end{array}$$

Fractional part:

$$\frac{1}{8} + \frac{4}{8^2} = \frac{12}{64} = \frac{3}{16}.$$

This is reduced to a radix-fraction in the scale of six as follows:

$$\begin{array}{r}
3 \\
\underline{6} \\
16)\,18\,(1 \\
\underline{16} \\
2 \\
\underline{6} \\
16)\,12\,(0 \\
\underline{6} \\
16)\,72\,(4 \\
\underline{64} \\
8 \\
\underline{6} \\
16)\,48\,(3 \\
\underline{48}
\end{array}$$

$\therefore$ the answer is 45.1043

Exercise 124.

1. If 6, 7, 8, 3, 2 are the digits of a number in the scale of r, beginning from the right, write the number.

2. Find the product of 234 and 125 when r is the base.

3. In what scale will 756 be expressed by 530?

4. In what scale will 540 be the square of 23?

5. In what scale will 212, 1101, 1220 be in arithmetical progression?

6. Multiply 31.24 by 0.31 in the scale of 5.

CHAPTER XXXII.

THEORY OF NUMBERS.

413. Definitions. In the present chapter, by *number* will be meant *positive integer*. The terms *prime*, *composite*, will be used in the ordinary arithmetical sense.

A *multiple* of a is a number which contains the factor a, and may be written ma.

An even number, since it contains the factor 2, may be written $2m$; an odd number may be written $2m+1$, $2m-1$, $2m+3$, $2m-3$, etc.

A number a is said *to divide* another number b when $\frac{b}{a}$ is an integer.

414. Resolution into Prime Factors. *A number can be resolved into prime factors in only one way.*

Let N be the number; suppose $N=abc \ldots\ldots$, where $a, b, c, \ldots\ldots$ are prime numbers; suppose also $N=\alpha\beta\gamma \ldots\ldots$ where $\alpha, \beta, \gamma, \ldots\ldots$ are prime numbers.

Then, $$abc \cdots\cdots = \alpha\beta\gamma \cdots\cdots$$

Hence, α must divide the product $abc \ldots\ldots$; but $a, b, c, \ldots\ldots$ are all prime numbers; hence α must be equal to some one of them, a suppose.

Dividing by a, $$bc \cdots\cdots = \beta\gamma \cdots\cdots,$$

and so on. Hence, the factors in $\alpha\beta\gamma \ldots\ldots$ are equal to those in $abc \ldots\ldots$, and the theorem is proved.

415. Divisibility of a Product. I. *If a number* **a** *divides a product* **bc**, *and is prime to* **b**, *it must divide* **c**.

For, since a divides bc, every prime factor of a must be found in bc; but since a is prime to b, no factor of a will be found in b; hence all the prime factors of a are found in c; that is, a divides c.

From this theorem it follows that:

II. *If a prime number* **a** *divides a product* **bcde....**, *it must divide some factor of that product; and conversely.*

III. *If a prime number divides* $\mathbf{b}^n$, *it must divide* **b**.

IV. *If* **a** *is prime to* **b** *and* **c**, *it is prime to* **bc**.

V. *If* **a** *is prime to* **b**, *every power of* **a** *is prime to every power of* **b**.

416. Theorem. *If* $\frac{\mathbf{a}}{\mathbf{b}}$, *a fraction in its lowest terms, is equal to another fraction* $\frac{\mathbf{c}}{\mathbf{d}}$, *then* **c** *and* **d** *are equimultiples of* **a** *and* **b**.

If $\frac{a}{b}=\frac{c}{d}$, then $\frac{ad}{b}=c$. Since b will not divide a, it must divide d; hence d is a multiple of b.

Let $d=mb$, m being an integer; since $\frac{a}{b}=\frac{c}{d}$, and $d=mb$, $\frac{a}{b}=\frac{c}{mb}$; therefore $c=ma$.

Hence, c and d are equimultiples of a and b.

From the above theorem, it follows that in the decimal scale of notation a common fraction in its lowest terms will produce a non-terminating decimal if its denominator contains any prime factor except 2 and 5.

For a terminating decimal is equivalent to a fraction with a denominator 10^n. Therefore, a fraction $\frac{a}{b}$ in its lowest terms cannot be equal to such a fraction, unless 10^n is a multiple of b. But 10^n, that is, $2^n \times 5^n$, contains no factors

besides 2 and 5, and hence cannot be a multiple of b, if b contains any factors except these.

417. Square Numbers. If a square number is resolved into its prime factors, the exponent of each factor will be even.

For, if $$N = a^p \times b^q \times c^r \cdots\cdots,$$
$$N^2 = a^{2p} \times b^{2q} \times c^{2r} \cdots\cdots$$

Conversely: A number which has the exponents of all its prime factors even will be a perfect square; therefore, to change any number to a perfect square,

Resolve the number into its prime factors, select the factors which have odd exponents, and multiply the given number by the product of these factors.

Thus, to find the least number by which 250 must be multiplied to make it a perfect square.

$250 = 2 \times 5^3$, in which 2 and 5 are the factors which have odd exponents.

Hence the multiplier required is $2 \times 5 = 10$.

418. Divisibility of Numbers.

I. *If two numbers* **N** *and* **N′**, *when divided by* **a**, *have the same remainder, their difference is divisible by* **a**.

For, if N when divided by a have a quotient q and a remainder r, then
$$N = qa + r.$$

And if N' when divided by a have a quotient q' and a remainder r, then
$$N' = q'a + r.$$

Therefore, $$N - N' = (q - q')a.$$

II. *If the difference of two numbers* **N** *and* **N′** *is divisible by* **a**, *then* **N** *and* **N′** *when divided by* **a** *will have the same remainder.*

For, if $$N - N' = (q - q')a,$$

then $$\frac{N}{a} - \frac{N'}{a} = q - q'.$$

Therefore, $$\frac{N}{a} - q = \frac{N'}{a} - q'.$$

That is, $$N - aq = N' - aq'.$$

III. *If two numbers* **N** *and* **N'**, *when divided by a given number* **a**, *have remainders* **r** *and* **r'**, *then* **NN'** *and* **rr'** *when divided by* **a** *will have the same remainder.*

For, if $$N = qa + r,$$

and $$N' = q'a + r',$$

then $$NN' = qq'a^2 + qar' + q'ar + rr'$$
$$= (qq'a + qr' + q'r)a + rr'.$$

Therefore, by II., NN' and rr' when divided by a will have the same remainder.

As a particular case, 37 and 47 when divided by 7 have remainders 2 and 5 respectively.

Now $37 \times 47 = 1739$ and $2 \times 5 = 10$.

The remainder, when each of these two numbers is divided by 7, is 3.

Note. From II. it follows that, in the scale of ten:

(1) A number is divisible by 2, 4, 8, ... if the numbers denoted by its last digit, last two digits, last three digits, ... are divisible respectively by 2, 4, 8, ...

(2) A number is divisible by 5, 25, 125, ... if the numbers denoted by its last digit, last two digits, last three digits, ... are divisible respectively by 5, 25, 125, ...

(3) If from a number the sum of its digits is subtracted, the remainder will be divisible by 9.

For, if from a number expressed in the form

$$a + 10b + 10^2c + 10^3d + \cdots\cdots$$

$$a + \quad b + \quad c + \quad d + \cdots\cdots \qquad \text{is subtracted,}$$

the remainder will be $(10-1)b + (10^2-1)c + (10^3-1)d + \cdots\cdots$ and $10-1$, 10^2-1, 10^3-1, ... will be a series of 9's.

Therefore, the remainder is divisible by 9.

(4) Hence, a number N may be expressed in the form

$$9n + s \text{ (if } s \text{ denotes the sum of its digits);}$$

and N is divisible by 3 if s is divisible by 3; and also by 9 if s is divisible by 9.

(5) A number is divisible by 11 if the difference between the sum of its digits in the even places and the sum of its digits in the odd places is 0 or a multiple of 11.

For, a number N expressed by digits (beginning from the right) a, b, c, d, ... may be put in the form of

$$N = a + 10b + 10^2c + 10^3d + \cdots\cdots$$

$$\therefore N - a + b - c + d - \cdots\cdots = (10+1)b + (10^2-1)c + (10^3+1)d + \cdots\cdots$$

But $10+1$ is a factor of $10+1$, 10^2-1, 10^3+1, ...

Therefore, $N - a + b - c + d - \cdots\cdots$ is divisible by $10 + 1 = 11$.

Hence, the number N may be expressed in the form

$$11n + (a + c + \cdots\cdots) - (b + d + \cdots\cdots),$$

and is a multiple of 11 if $(a+c+\cdots\cdots)-(b+d+\cdots\cdots)$ is 0 or a multiple of 11.

419. Theorem. *The product of* **r** *consecutive integers is divisible by* $\underline{|r}$.

Represent by $P_{n,k}$ the product of k consecutive integers beginning with n.

Then, $\qquad P_{n,k} = n(n+1)\cdots\cdots(n+k-1);$

$$\begin{aligned} P_{n+1,k+1} &= (n+1)(n+2)\cdots\cdots(n+k)(n+k+1) \\ &= n(n+1)(n+2)\cdots\cdots(n+k) \\ &\qquad + (k+1)(n+1)(n+2)\cdots\cdots(n+k). \end{aligned}$$

$$\therefore P_{n+1,k+1} = P_{n,k+1} + (k+1)P_{n+1,k}.$$

Assume, for the moment, that the product of any k consecutive integers is divisible by $\underline{|k}$.

Then, $$P_{n+1,\,k+1} = P_{n,\,k+1} + (k+1)\,M\underline{|k};$$
or $$P_{n+1,\,k+1} = P_{n,\,k+1} + M\underline{|k+1};$$
where M is an integer.

From this it is seen that if $P_{n,\,k+1}$ is divisible by $\underline{|k+1}$, $P_{n+1,\,k+1}$ is also divisible by $\underline{|k+1}$; but $P_{1,\,k+1}$ is divisible by $\underline{|k+1}$ since $P_{1,\,k+1} = \underline{|k+1}$. $\therefore$ $P_{2,\,k+1}$ is divisible by $\underline{|k+1}$; $\therefore$ $P_{3,\,k+1}$ is divisible by $\underline{|k+1}$; and so on.

Hence, the product of any $k+1$ consecutive integers is divisible by $\underline{|k+1}$, if it is known that the product of any k consecutive integers is divisible by $\underline{|k}$. But the product of any 2 consecutive integers is divisible by $\underline{|2}$; therefore, the product of any 3 consecutive integers is divisible by $\underline{|3}$; therefore, the product of any 4 consecutive integers is divisible by $\underline{|4}$; and so on. Therefore, the product of any r consecutive integers is divisible by $\underline{|r}$.

Exercise 125.

Find the least number by which each of the following numbers must be multiplied in order that the product may be a square number:

1. 2625. **2.** 3675. **3.** 4374. **4.** 74088.

5. If m and n are positive integers, both odd or both even, show that $m^2 - n^2$ is divisible by 4.

6. Show that $n^2 - n$ is always even.

7. Show that $n^3 - n$ is divisible by 6 if n is even; and by 24 if n is odd.

CHAPTER XXXIII.

VARIABLES AND LIMITS.

420. Constants and Variables. A number that, under the conditions of the problem into which it enters, may be made to assume any one of an unlimited number of values is called a **variable.**

A number that, under the conditions of the problem into which it enters, has a fixed value is called a **constant.**

Variables are generally represented by x, y, z, etc.; constants, by the Arabic numerals, and by a, b, c, etc.

421. Functions. Two variables may be so related that a change in the value of one produces a change in the value of the other. In this case one variable is said to be a **function** of the other.

Thus, if a man walks on a road at a uniform rate of a miles per hour, the number of miles he walks and the number of hours he walks are both variables, and the first is a function of the second. If y be the number of miles he has walked at the end of x hours, y and x are connected by the relation $y = ax$, and y is a function of x. Also $x = \frac{y}{a}$; hence, x is also a function of y.

When one of two variables is a function of the other, the relation between them is generally expressed by an equation. If a value of one variable is assumed, the corresponding value of the other variable can be found from the given equation of relation between the two variables.

The variable of which the value is assumed is called the *independent* variable; the variable of which the value is

found from the given relation of the two variables is called the *dependent* variable.

In the last example we may assume values of x, and find the corresponding values of y from the relation $y = ax$; or assume values of y, and find the corresponding values of x from the relation $x = \frac{y}{a}$. In the first case x is the independent variable, and y the dependent; in the second case y is the independent variable, and x the dependent.

422. Limits. As a variable changes its value, it may approach some constant; if the variable can be made to approach the constant *as near as we please*, but cannot be made *absolutely equal to the constant*, the variable is said to approach the constant *as a limit*, and the constant is called the **limit** of the variable.

Let x represent the sum of n terms of the infinite series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots;$$

then (§ 314),
$$x = \frac{(\frac{1}{2})^n - 1}{\frac{1}{2} - 1} = \frac{2^n - 1}{2^{n-1}} = 2 - \frac{1}{2^{n-1}}.$$

Suppose n to increase; then, $\frac{1}{2^{n-1}}$ decreases, and x approaches 2.

Since we can take as many terms of the series as we please, n can be made as large as we please; therefore, $\frac{1}{2^{n-1}}$ can be made as small as we please, and x can be made to approach 2 as near as we please.

We cannot, however, make x absolutely equal to 2.

If we take any *assigned* value, as $\frac{1}{10000}$, we can make the difference between 2 and x less than this assigned value; for we have only to take n so large that $\frac{1}{2^{n-1}}$ is less than $\frac{1}{10000}$; that is, that 2^{n-1} is greater than 10,000: this will be accomplished by taking n as large as 15. Similarly, by taking n large enough, we can make the difference between 2 and x less than *any* assigned value.

Since $2 - x$ can be made as small as we please, it follows that the sum of n terms of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$, as n is constantly increased, approaches 2 *as a limit.*

423. Test for a Limit. In order to prove that a variable approaches a constant as a limit, it is *necessary* and *sufficient* to prove that the difference between the variable and the constant can be made *as near to zero as we please*, but cannot be made *absolutely equal to zero.*

A variable may approach a constant without approaching it *as a limit.* Thus, in the last example x approaches 3, but not as a limit; for $3 - x$ cannot be made as near to 0 as we please, since it cannot be made less than 1.

424. Infinites. As a variable changes its value, it may constantly increase in numerical value; if the variable can become numerically greater than any assigned value, *however great* this assigned value may be, the variable is said to *increase without limit*, or to *increase indefinitely.*

When a variable is conceived to have a value greater than any assigned value, however great this assigned value may be, the variable is said to become *infinite;* such a variable is called an *infinite* number, or simply an **infinite.**

425. Infinitesimals. As a variable changes its value, it may constantly decrease in numerical value; if the variable can become numerically less than any assigned value, *however small* this assigned value may be, the variable is said to *decrease without limit*, or to *decrease indefinitely.*

In this case the variable approaches 0 as a limit.

When a variable which approaches 0 as a limit is conceived to have a value less than any assigned value, however small this assigned value may be, the variable is said to become *infinitesimal;* such a variable is called an *infinitesimal number*, or simply an **infinitesimal.**

426. Infinites and infinitesimals are *variables*, not constants. There is no idea of *fixed* value implied in either an infinite or an infinitesimal.

An infinitesimal is not 0. An infinitesimal is a variable arising from the division of a quantity into a constantly increasing number of parts; 0 is a constant arising from taking the difference of two equal quantities.

A number which cannot become infinite is said to be **finite.**

427. Relations between Infinites and Infinitesimals.

I. *If* $\mathbf{x}$ *is infinitesimal, and* $\mathbf{a}$ *is finite and not* 0, *then* $\mathbf{ax}$ *is infinitesimal.* For, ax can be made as small as we please since x can be made as small as we please.

II. *If* $\mathbf{X}$ *is infinite, and* $\mathbf{a}$ *is finite and not* 0, *then* $\mathbf{aX}$ *is infinite.* For aX can be made as large as we please since X can be made as large as we please.

III. *If* $\mathbf{x}$ *is infinitesimal, and* $\mathbf{a}$ *is finite and not* 0, *then* $\frac{\mathbf{a}}{\mathbf{x}}$ *is infinite.* For $\frac{a}{x}$ can be made as large as we please since x can be made as small as we please.

IV. *If* $\mathbf{X}$ *is infinite, and* $\mathbf{a}$ *is finite and not* 0, *then* $\frac{\mathbf{a}}{\mathbf{X}}$ *is infinitesimal.* For $\frac{a}{X}$ can be made as small as we please since X can be made as large as we please.

In the above theorems a may be a constant or a variable; the only restriction on the value of a is that it shall not become either infinite or zero.

428. Abbreviated Notation. An infinite is often represented by ∞. In § 427, III. and IV. are sometimes written:

$$\frac{a}{0}=\infty, \quad \frac{a}{\infty}=0.$$

The expression $\frac{a}{0}$ cannot be interpreted literally, since we cannot divide by 0: and the expression $\frac{a}{\infty}=0$ cannot be interpreted liter-

ally, since we can find no number such that the quotient obtained by dividing a by that number is zero.

$\frac{a}{0}=\infty$ is simply an abbreviated way of writing: *if* $\frac{a}{x}=X$, *and* x *approaches* 0 *as a limit,* X *increases without limit.* $\frac{a}{\infty}=0$ is simply an abbreviated way of writing: *if* $\frac{a}{X}=x$, *and* X *increases without limit,* x *approaches* 0 *as a limit.*

429. Approach to a Limit. When a variable approaches a limit, it may approach its limit in one of three ways:

(1) The variable may be always less than its limit.

(2) The variable may be always greater than its limit.

(3) The variable may be sometimes less and sometimes greater than its limit.

If x represents the sum of n terms of the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\cdots$, x is always less than its limit 2.

If x represents the sum of n terms of the series $3-\frac{1}{2}-\frac{1}{4}-\frac{1}{8}-\cdots\cdots$, x is always greater than its limit 2.

If x represents the sum of n terms of the series $3-\frac{3}{2}+\frac{3}{4}-\frac{3}{8}+\cdots\cdots$, we have (§ 314)

$$x=\frac{3-3(-\frac{1}{2})^n}{1+\frac{1}{2}}=2-2(-\tfrac{1}{2})^n.$$

As n is indefinitely increased, x evidently approaches 2 as a limit.

If n is even, x is less than 2; if n is odd, x is greater than 2. Hence, if n be increased by taking each time one more term, x will be alternately less than and greater than 2. If, for example,

$$\begin{array}{lllllll} n= & 2, & 3, & 4, & 5, & 6, & 7, \\ x= & 1\frac{1}{2}, & 2\frac{1}{4}, & 1\frac{7}{8}, & 2\frac{1}{16}, & 1\frac{31}{32}, & 2\frac{1}{64} \end{array}$$

In whatever way a variable approaches its limit, the test of § 423 always applies.

430. Equal Variables. *If two variables are equal and are so related that a change in the one produces such a change*

in the other that they continue equal, and each approaches a limit, then their limits are equal.

Let x and y be the variables, a and b their respective limits. To prove $a=b$. We have (§ 423)

$$a=x+x',\ b=y+y',$$

where x' and y' are variables which approach 0 as a limit.

Then, since the equation $x=y$ always holds, we have, by subtraction, $$a-b=x'-y'.$$

$x'-y'$ can be made less than any assigned value since x' and y' can each be made less than any assigned value.

Since $x'-y'$ is always equal to the constant $a-b$, $x'-y'$ must be a constant. But the only constant which is less than any assigned value is 0. Therefore $x'-y'=0$, and hence $a-b=0$. $\therefore\ a=b$.

431. Limit of a Sum. *The limit of the algebraic sum of any finite number of variables is the algebraic sum of their respective limits.*

Let $x, y, z, \ldots\ldots,$ be variables;

$a, b, c, \ldots\ldots,$ their respective limits.

Then $a-x,\ b-y,\ c-z,\ \ldots\ldots,$ are variables which can each be made less than any assigned value (§ 423).

Then $(a-x)+(b-y)+(c-z)+\cdots\cdots$ can be made less than any assigned value.

For, let v be the numerically greatest of the variables $a-x$, $b-y$, $c-z$, ..., and n the number of variables.

Then, $(a-x)+(b-y)+(c-z)+\cdots\cdots < v+v+v\cdots\cdots$ to n terms

$$< nv;$$

but nv can be made less than any assigned value since n is finite and v can be made less than any assigned value (§ 427, I.).

Therefore, $(a-x)+(b-y)+(c-z)$ ....., which is less than nv, can be made less than any assigned value.

$\therefore\ (a+b+c+\cdots\cdots)-(x+y+z+\cdots\cdots)$ can be made less than any assigned value.

$\therefore\ a+b+c+\cdots\cdots$ is the limit of $(x+y+z+\cdots\cdots)$. § 423

432. Limit of a Product. *The limit of the product of two or more variables is the product of their respective limits.*

Let x and y be variables, a and b their respective limits.

To prove that ab is the limit of xy.

Put $x=a-x'$, $y=b-y'$; then x' and y' are variables which can be made less than any assigned value (§ 423).

Now,
$$xy=(a-x')(b-y')$$
$$=ab-ay'-bx'+x'y'.$$
$$\therefore\ ab-xy=ay'+bx'-x'y'.$$

Since every term on the right contains x' or y', the whole right member can be made less than any assigned value (§ 427, I.). Hence, $ab-xy$ can be made less than any assigned value.

$\therefore\ ab$ is the limit of xy (§ 423).

Similarly for three or more variables.

433. Limit of a Quotient. *The limit of the quotient of two variables is the quotient of their limits.*

Let x and y be variables, a and b their respective limits.

Put $a-x=x'$, and $b-y=y'$; then x' and y' are variables with limit 0 (§ 423).

We have $x=a-x'$, $y=b-y'$, and $\dfrac{x}{y}=\dfrac{a-x'}{b-y'}$.

Now
$$\frac{a}{b}-\frac{x}{y}=\frac{a}{b}-\frac{a-x'}{b-y'}=\frac{bx'-ay'}{b(b-y')}.$$

The numerator of the last expression approaches 0 as a limit, and the denominator approaches b^2; hence, the expression approaches 0 as a limit (§ 427, I.).

$\therefore\ \dfrac{a}{b}-\dfrac{x}{y}$ approaches 0 as a limit. $\therefore\ \dfrac{a}{b}$ is the limit of $\dfrac{x}{y}$.

434. Vanishing Fractions. When one or more variables are involved in both numerator and denominator of a fraction, it may happen that for certain values of the variables both numerator and denominator of the fraction vanish. The fraction then assumes the form $\frac{0}{0}$, which is a form without meaning; as even the interpretation of § 428 fails, since the numerator is 0. If, however, there is but *one* variable involved, we may obtain a value as follows:

Let x be the variable, and a the value of x for which the fraction assumes the form $\frac{0}{0}$. Give to x a value a little greater than a, as $a+z$; the fraction will now have a definite value. Find the limit of this last value as z is indefinitely decreased. This limit is called the **limiting value** of the fraction.

(1) Find the limiting value of $\dfrac{x^2-a^2}{x-a}$ as x approaches a.

When x has the value a, the fraction assumes the form $\frac{0}{0}$.

Put $x=a+z$; the fraction becomes

$$\frac{(a+z)^2-a^2}{(a+z)-a}=\frac{2az+z^2}{z}.$$

Since z is not 0, we can divide by z and obtain $2a+z$.

As z is indefinitely decreased, this approaches $2a$ as a limit. Hence $2a$ is the answer required.

(2) Find the limiting value of $\dfrac{2x^3-4x+5}{3x^3+2x-1}$ when x becomes infinite.

We have
$$\frac{2x^3-4x+5}{3x^3+2x^2-1}=\frac{2-\dfrac{4}{x^2}+\dfrac{5}{x^3}}{3+\dfrac{2}{x}-\dfrac{1}{x^3}}.$$

As x increases indefinitely, $\frac{1}{x}$ approaches 0, and the fraction approaches $\frac{2}{3}$.

Exercise 126.

Find the limiting values of:

1. $\dfrac{(4x^2-3)(1-2x)}{7x^3-6x+4}$ when x becomes infinitesimal.

2. $\dfrac{(x^2-5)(x^2+7)}{x^4+35}$ when x becomes infinite.

3. $\dfrac{(x+2)^3}{x^2+4}$ when x becomes infinitesimal.

4. $\dfrac{x^2-8x+15}{x^2-7x+12}$ when x approaches 3.

5. $\dfrac{x^2-9}{x^2+9x+18}$ when x approaches -3.

6. $\dfrac{x(x^2+4x+3)}{x^3+3x^2+5x+3}$ when x approaches -1.

7. $\dfrac{x^3+x^2-2}{x^3+2x^2-2x-1}$ when x approaches 1.

8. $\dfrac{4x+\sqrt{x-1}}{2x-\sqrt{x+1}}$ when x approaches 1.

9. $\dfrac{x-1}{\sqrt{x^2-1}+\sqrt{x-1}}$ when x approaches 1.

10. $\dfrac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}$ when x approaches 2.

11. $\dfrac{\sqrt{x-a}+\sqrt{x}-\sqrt{a}}{\sqrt{x^2-a^2}}$ when x approaches a.

12. If x approaches a as a limit, and n is a positive integer, show that the limit of x^n is a^n.

13. If x approaches a as a limit, and a is not 0, show that the limit of x^n is a^n, where n is a negative integer.

CHAPTER XXXIV.

SERIES.

435. Convergency of Series. For an infinite series to be convergent (§ 325) it is *necessary* and *sufficient* that the sum of all the terms after the nth, as n is indefinitely increased, should approach 0 as a limit.

Although each of the terms after the nth may approach 0 as a limit, their sum may not approach 0 as a limit.

Thus, take the harmonical series,

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \cdots\cdots\ \frac{1}{n},\ \frac{1}{n+1},\ \frac{1}{n+2},\ \cdots\cdots$$

Each term after the nth approaches 0 as n increases.

The sum of n terms after the nth term is

$$\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots\cdots+\frac{1}{2n},$$

which is $>\frac{1}{2n}+\frac{1}{2n}+\cdots\cdots$ to n terms; therefore $>n\times\frac{1}{2n}$; that is, $>\frac{1}{2}$.

Now, the first term is 1, the second term is $\frac{1}{2}$, the sum of the next two terms is greater than $\frac{1}{2}$, the sum of the succeeding four terms is greater than $\frac{1}{2}$; and so on. So that, by increasing n indefinitely, the sum will become greater than any finite multiple of $\frac{1}{2}$.

Therefore, the series is *divergent*.

To determine whether the following series is convergent:

$$1+\frac{1}{1}+\frac{1}{2!}+\frac{1}{3!}+\cdots\cdots\frac{1}{(n-1)!}+\frac{1}{n!}+\frac{1}{(n+1)!}+\cdots\cdots$$

The nth term is $\frac{1}{(n-1)!}$. The sum of the remaining terms is

$$\frac{1}{n!}+\frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\cdots\cdots=\frac{1}{n!}\left(1+\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\cdots\cdots\right).$$

This is $<\frac{1}{\lfloor n}\left(1+\frac{1}{n}+\frac{1}{n^2}+\cdots\right)$; therefore, since $1+\frac{1}{n}+\frac{1}{n^2}+\cdots$ is the expansion of $\frac{1}{1-\frac{1}{n}}$,

$$\frac{1}{\lfloor n}\left(\frac{1}{1-\frac{1}{n}}\right), \text{ or } \frac{1}{\lfloor n}\left(\frac{n}{n-1}\right); \text{ that is, } <\frac{1}{(n-1)\lfloor n-1}.$$

But as n increases indefinitely, this last expression approaches 0 as a limit. Hence, the series is *convergent.*

436. Test for Convergency of a Series. *If the terms of an infinite series are all positive, and the limit of the nth term is* 0, *then if the limit of the ratio of the* $(n+1)$*th term to the nth term, as n is indefinitely increased, is less than* 1, *the series is convergent.*

Let $u_1, u_2, u_3, \ldots\ldots u_n, u_{n+1}, u_{n+2}, \ldots\ldots$ be an infinite series.

Let r represent the limit of the ratio $\frac{u_{n+1}}{u_n}$ as n increases indefinitely, and suppose r to be positive and less than 1.

Let k be some *fixed* number between r and 1, and take k so near 1 that $\frac{u_{n+1}}{u_n}$, $\frac{u_{n+2}}{u_{n+1}}$, ..., shall each be $<k$.

$$\text{Then, } \frac{u_{n+1}}{u_n}<k, \qquad \frac{u_{n+2}}{u_{n+1}}<k, \qquad \frac{u_{n+3}}{u_{n+2}}<k, \ldots\ldots$$

$$\therefore\ u_{n+1}<ku_n, \qquad u_{n+2}<ku_{n+1}, \qquad u_{n+3}<ku_{n+2}, \ldots\ldots$$

$$\therefore\ u_{n+1}<ku_n, \qquad u_{n+2}<k^2u_n, \qquad u_{n+3}<k^3u_n, \ldots\ldots$$

$$\therefore\ u_{n+1}+u_{n+2}+u_{n+3}+\cdots\cdots<u_n(k+k^2+k^3+\cdots\cdots)$$

$$\therefore\ u_{n+1}+u_{n+2}+u_{n+3}+\cdots\cdots<u_n\frac{k}{1-k},$$

since $\quad k+k^2+k^3+\cdots\cdots$ is the expansion of $\frac{k}{1-k}$.

But, by hypothesis, u_n approaches 0 as a limit as n is indefinitely increased. Hence, the series is *convergent.*

Similarly, when r is negative, and between 0 and -1.

Thus, in the series

$$1 + \frac{1}{1} + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \cdots\cdots \frac{1}{\underline{|n-1}} + \frac{1}{\underline{|n}} + \cdots\cdots,$$

$\frac{u_{n+1}}{u_n} = \frac{1}{n}$, and this approaches 0 as a limit as n is indefinitely increased; moreover, the nth term, $\frac{1}{\underline{|n-1}}$, approaches 0 as a limit.

Hence, the series is convergent.

If $r > 1$, there must be in the series some term from which the succeeding term is greater than the next preceding term; so that the remaining terms will form an increasing series, and therefore the series is not convergent.

If $r = \pm 1$, this value gives no information as to whether the series is convergent or not; and in such cases other tests must be applied.

If $r < 1$, but approaches 1, or -1, as a limit, then no fixed value k can be found which will always lie between r and ± 1, and other tests of convergency must be applied.

Thus, in the infinite series

$$\frac{1}{1^m} + \frac{1}{2^m} + \frac{1}{3^m} + \cdots\cdots + \frac{1}{n^m} + \frac{1}{(n+1)^m} + \cdots\cdots,$$

r, the ratio of the $(n+1)th$ term to the nth term, is

$$\left(\frac{n}{n+1}\right)^m = \left(1 - \frac{1}{n+1}\right)^m,$$

which approaches 1 as a limit as n increases.

Suppose m positive and greater than 1; then the first term of the series is 1. The sum of the next two terms is less than $\frac{2}{2^m}$. The sum of the next four terms is less than $\frac{4}{4^m}$. The sum of the next eight terms is less than $\frac{8}{8^m}$; and so on. Hence, the sum of the series is less than $1 + \frac{2}{2^m} + \frac{4}{4^m} + \frac{8}{8^m} + \cdots\cdots$, or $< 1 + \frac{1}{2^{m-1}} + \frac{1}{4^{m-1}} + \frac{1}{8^{m-1}} + \cdots\cdots$, which is evidently convergent when m is positive and greater than 1.

If m is positive and equal to 1, the given series becomes

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots\cdots,$$

which is the harmonical series shown in § 435 to be divergent.

If m is negative, or less than 1, each term of the series is then greater than the corresponding term in the harmonical series, and hence the series is divergent.

437. Special Case. *If the terms of an infinite series are alternately positive and negative; if, also, the terms continually decrease, and the limit of the nth term is zero, then the series is convergent.*

Consider the infinite series,

$$u_1 - u_2 + u_3 - u_4 + \cdots\cdots \mp u_n \pm u_{n+1} \mp u_{n+2} \pm \cdots\cdots$$

The sum of the terms after the *nth* term is

$$\pm [u_{n+1} - (u_{n+2} - u_{n+3}) - (u_{n+4} - u_{n+5}) - \cdots\cdots],$$

which may be written

$$\pm [(u_{n+1} - u_{n+2}) + (u_{n+3} - u_{n+4}) + (u_{n+5} - u_{n+6}) + \cdots\cdots].$$

Since the terms are continually diminishing, each of the groups in either form of expression is positive, and therefore the absolute value of the required sum is seen, from the first form of expression, to be less than u_{n+1}; and from the second form of expression, to be greater than $u_{n+1} - u_{n+2}$. But both u_{n+1} and u_{n+2} approach zero as n increases indefinitely; therefore the sum of the series after the *nth* term approaches zero, and the series is convergent.

In finding the sum of an infinite decreasing series of which the terms are alternately positive and negative, if we stop at any term, the error will be less than the next succeeding term.

The series $1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \cdots\cdots \pm \dfrac{1}{n} \mp \dfrac{1}{n+1} \pm \cdots\cdots$ is convergent.

For, we may write the series

$$(1-\tfrac{1}{2})+(\tfrac{1}{3}-\tfrac{1}{4})+(\tfrac{1}{5}-\tfrac{1}{6})+\cdots\cdots, \text{ or } 1-(\tfrac{1}{2}-\tfrac{1}{3})-(\tfrac{1}{4}-\tfrac{1}{5})-\cdots\cdots,$$

which shows that its sum is greater than $\frac{1}{2}$, and less than 1.

Observe that the present test applies to series in which $\frac{u_{n+1}}{u_n}$ approaches 1, or -1, as a limit. To such series the test of § 436 will not apply.

438. Convergency of the Binomial Series. In the expansion of $(1+x)^n$, the ratio of the $(r+1)$th term to the rth term is (§ 340)

$$\frac{n-r+1}{r}x, \text{ or } \left(\frac{n+1}{r}-1\right)x.$$

If x is positive and r greater than $n+1$, the expression $\frac{n+1}{r}-1$ is negative; hence the terms in which r is greater than $n+1$ are alternately positive and negative.

If x is negative, the terms in which r is greater than $n+1$ are all positive. In either case we have

$$\frac{u_{r+1}}{u_r}=\left(\frac{n+1}{r}-1\right)x;$$

as r is indefinitely increased, this approaches the limit $-x$. Hence (§ 436), the series is convergent if x is numerically less than 1.

If n is fractional or negative, the expansion of $(a+b)^n$ must be in the form $a^n\left(1+\frac{b}{a}\right)^n$ if $a>b$; and in the form $b^n\left(1+\frac{a}{b}\right)$ if $b>a$ (§ 344).

439. Examples.

(1) For what values of x is the infinite series

$$x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\cdots\pm\frac{x^n}{n}\mp\cdots\cdots \text{ convergent?}$$

Here, $$r = \frac{u_{n+1}}{u_n} = \left(\frac{n}{n+1}\right)x = \left(1 - \frac{1}{n+1}\right)x.$$

As n is indefinitely increased, r approaches x as a limit. Hence, the series is convergent when x is numerically less than 1; and divergent when x is numerically greater than 1.

When $x = 1$, the series is convergent by § 437.

When $x = -1$, the series becomes

$$-\left(1 + \frac{1}{2} + \frac{1}{3} \cdots\cdots + \frac{1}{n} + \cdots\cdots\right),$$

the harmonical series already shown to be divergent (§ 435).

(2) For what values of x is the infinite series

$$\frac{x}{1 \times 2} + \frac{x^2}{2 \times 3} + \frac{x^3}{3 \times 4} + \cdots\cdots \frac{x^n}{n(n+1)}$$ convergent?

Here, $$r = \frac{u_{n+1}}{u_n} = \left(\frac{n}{n+2}\right)x = \left(\frac{1}{1 + \frac{2}{n}}\right)x.$$

As n is indefinitely increased, r approaches x as a limit.

If x is numerically less than 1, the series is convergent.

If x is numerically greater than 1, the series is divergent.

If $x = 1$, every term of the series

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots\cdots$$

is less than the corresponding term of the series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\cdots$$

This last series is a special case of the series

$$\frac{1}{1^m} + \frac{1}{2^m} + \frac{1}{3^m} + \cdots\cdots$$

and is therefore convergent (§ 436).

Hence, $$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots\cdots$$ is convergent.

If $x = -1$, the series becomes

$$-\frac{1}{1 \times 2} + \frac{1}{2 \times 3} - \frac{1}{3 \times 4} \cdots\cdots$$

and is convergent by § 437.

Series of Differences.

440. Definitions. If, in any series, we subtract from each term the preceding term, we obtain a first series of differences; in like manner from this last series we may obtain a second series of differences; and so on. In an arithmetical series the second differences all vanish.

There are series, allied to arithmetical series, in which not the first, but the second, or third, etc., differences vanish.

Thus take the series

$$\begin{array}{llllllllllllllll}
 & 1 & & 5 & & 12 & & 24 & & 43 & & 71 & & 110 & & \ldots\ldots \\
\text{1st differences,} & & 4 & & 7 & & 12 & & 19 & & 28 & & 39 & & \ldots\ldots & \\
\text{2d differences,} & & & 3 & & 5 & & 7 & & 9 & & 11 & & \ldots\ldots & & \\
\text{3d differences,} & & & & 2 & & 2 & & 2 & & 2 & & \ldots\ldots & & & \\
\text{4th differences,} & & & & & 0 & & 0 & & 0 & & \ldots\ldots & & & &
\end{array}$$

In general, if a_1, a_2, a_3, ... be such a series,

b_1, b_2, b_3, ... be the first differences,

c_1, c_2, c_3, ... be the second differences,

d_1, d_2, d_3, ... be the third differences,

e_1, e_2, e_3, ... be the fourth differences,

$$\begin{array}{llllllllllllllll}
\text{we have} & a_1 & & a_2 & & a_3 & & a_4 & & a_5 & & a_6 & & a_7 & & \ldots\ldots \\
\text{1st differences,} & & b_1 & & b_2 & & b_3 & & b_4 & & b_5 & & b_6 & & \ldots\ldots & \\
\text{2d differences,} & & & c_1 & & c_2 & & c_3 & & c_4 & & c_5 & & \ldots\ldots & & \\
\text{3d differences,} & & & & d_1 & & d_2 & & d_3 & & d_4 & & \ldots\ldots & & & \\
\text{4th differences,} & & & & & e_1 & & e_2 & & e_3 & & \ldots\ldots & & & &
\end{array}$$

and finally arrive at differences which all vanish.

441. Any Required Term. Let us take a series in which the fifth series of differences vanishes. Any other case can be treated in a manner precisely similar. From the way in which the successive series are formed, we have:

$$
\begin{array}{ll}
a_2 = a_1 + b_1 & a_3 = a_2 + b_2 = a_1 + 2b_1 + c_1 \\
b_2 = b_1 + c_1 & b_3 = b_2 + c_2 = b_1 + 2c_1 + d_1 \\
c_2 = c_1 + d_1 & c_3 = c_2 + d_2 = c_1 + 2d_1 + e_1 \\
d_2 = d_1 + e_1 & d_3 = d_2 + e_2 = d_1 + 2e_1 \\
e_2 = e_1 & e_3 = e_2 \qquad = e_1
\end{array}
$$

$$
\begin{aligned}
a_4 &= a_3 + b_3 = a_1 + 3b_1 + 3c_1 + d_1 \\
b_4 &= b_3 + c_3 = b_1 + 3c_1 + 3d_1 + e_1 \\
c_4 &= c_3 + d_3 = c_1 + 3d_1 + 3e_1 \\
d_4 &= d_3 + e_3 = d_1 + 3e_1
\end{aligned}
$$

$$
\begin{aligned}
a_5 &= a_4 + b_4 = a_1 + 4b_1 + 6c_1 + 4d_1 + e_1 \\
b_5 &= b_4 + c_4 = b_1 + 4c_1 + 6d_1 + 4e_1 \\
c_5 &= c_4 + d_4 = c_1 + 4d_1 + 6e_1
\end{aligned}
$$

$$
\begin{aligned}
a_6 &= a_5 + b_5 = a_1 + 5b_1 + 10c_1 + 10d_1 + 5e_1 \\
b_6 &= b_5 + c_5 = b_1 + 5c_1 + 10d_1 + 10e_1
\end{aligned}
$$

$$a_7 = a_6 + b_6 = a_1 + 6b_1 + 15c_1 + 20d_1 + 15e_1$$

and so on.

The student will observe that the coefficients in the expression for a_5 are those of the expansion of $(x+y)^4$, and similarly for a_6 and a_7; hence, in general, if we represent a_1, b_1, c_1, etc., by a, b, c, etc., we have, putting for the $(n+1)$th term a_{n+1}, the formula

$$a_{n+1} = a + nb + \frac{n(n-1)}{1 \times 2}c + \frac{n(n-1)(n-2)}{1 \times 2 \times 3}d + \cdots\cdots$$

Ex. Find the 11th term of 1, 5, 12, 24, 43, 71, 110, ...

Here (§ 440) $a = 1$, $b = 4$, $c = 3$, $d = 2$, $e = 0$; and $n = 10$.

$$
\begin{aligned}
\therefore\ a_{11} &= a + 10b + 45c + 120d \\
&= 1 + 40 + 135 + 240 = 416.
\end{aligned}
$$

442. Sum of the Series. Form a new series of which the first term is 0, and the first series of differences $a_1, a_2, a_3, \ldots\ldots$ This series will be the following:

$$0,\ a_1,\ a_1+a_2,\ a_1+a_2+a_3,\ a_1+a_2+a_3+a_4,\ \ldots\ldots$$

The $(n+1)$th term of this series will be the sum of n terms of the series $a_1, a_2, a_3, \ldots\ldots$

Find the sum of 11 terms of the series 1, 5, 12, 24, 43, 71, ...

The new series is	0	1	6	18	42	85	156
First differences,	1	5	12	24	43	71	
Second differences,	4	7	12	19	28		
Third differences,	3	5	7	9			
Fourth differences,	2	2	2				

Here $a=0$, $b=1$, $c=4$, $d=3$, $e=2$; and $n=11$.

$$\begin{aligned}\therefore\ s &= a+11b+55c+165d+330e\\ &= 11+220+495+660\\ &= 1386.\end{aligned}$$

If s is the sum of n terms of the series $a_1, a_2, a_3, \ldots\ldots$

$$s=0+na+\frac{n(n-1)}{1\times 2}b+\frac{n(n-1)(n-2)}{1\times 2\times 3}c+\cdots\cdots$$

Ex. Find the sum of the squares of the first n natural numbers, $1^2, 2^2, 3^2, 4^2, \ldots\ldots n^2$.

Given series,	1	4	9	16	25	.....	n^2
First differences,	3	5	7	9	.....		
Second differences,	2	2	2	.....			
Third differences,	0	0	.....				

Therefore, $a=1$, $b=3$, $c=2$, $d=0$.

These values substituted in the general formula give

$$\begin{aligned}s &= n+\frac{n(n-1)}{1\times 2}\times 3+\frac{n(n-1)(n-2)}{1\times 2\times 3}\times 2\\ &= n\left\{1+\frac{3n}{2}-\frac{3}{2}+\frac{1}{3}(n^2-3n+2)\right\}\\ &= \frac{n}{6}\{6+9n-9+2n^2-6n+4\}\\ &= \frac{n}{6}\{2n^2+3n+1\}=\frac{n(n+1)(2n+1)}{6}.\end{aligned}$$

443. Piles of Spherical Shot. I. When the pile is in the form of a triangular pyramid, the summit consists of a single shot resting on three below; and these three rest on a course of six; and these six on a course of ten, and so on, so that the courses will form the series,

$$1,\ 1+2,\ 1+2+3,\ 1+2+3+4,\ \ldots\ 1+2+\cdots+n.$$

Given series,	1		3		6		10		15		.....			
First differences,		2		3		4		5		.....				
Second differences,			1		1		1		.....					
Third differences,				0		0		.....						

Here, $a = 1,\ b = 2,\ c = 1,\ d = 0.$

These values substituted in the general formula give

$$s = n + \frac{n(n-1)}{2} \times 2 + \frac{n(n-1)(n-2)}{2 \times 3}$$

$$= n\left\{1 + n - 1 + \frac{n^2 - 3n + 2}{6}\right\}$$

$$= \frac{n}{6}\{(n+1)(n+2)\}$$

$$= \frac{n(n+1)(n+2)}{1 \times 2 \times 3},$$

in which n is the number of balls in the side of the bottom course, or the number of courses.

II. When the pile is in the form of a pyramid with a square base, the summit consists of one shot, the next course consists of four balls, the next of nine, and so on. The number of shot, therefore, is the sum of the series,

$$1^2,\ 2^2,\ 3^2,\ 4^2,\ \ldots\ n^2.$$

Which, by § 442, is

$$\frac{n(n+1)(2n+1)}{1 \times 2 \times 3},$$

in which n is the number of balls in the side of the bottom course, or the number of courses.

III. When the pile has a base which is rectangular, but not square, the pile will terminate with a single row. Suppose p the number of shot in this row; then the second course will consist of $2(p+1)$ shot; the third course of $3(p+2)$; and the nth course of $n(p+n-1)$. Hence the series will be

$$p,\ 2p+2,\ 3p+6,\ \ldots\ldots,\ n(p+n-1).$$

Given series,	p	$2p+2$	$3p+6$	$4p+12$	.....
First differences,	$p+2$	$p+4$	$p+6$	.....	
Second differences,	2	2	.....		
Third differences,	0	.....			

Here, $a=p,\ b=p+2,\ c=2,\ d=0.$

These values substituted in the general formula give

$$s=np+\frac{n(n-1)}{2}(p+2)+\frac{n(n-1)(n-2)}{1\times 2\times 3}\times 2.$$

$$=\frac{n}{6}\{6p+3(n-1)(p+2)+2(n-1)(n-2)\}$$

$$=\frac{n}{6}(6p+3np-3p+6n-6+2n^2-6n+4)$$

$$=\frac{n}{6}(3np+3p+2n^2-2)$$

$$=\frac{n}{6}(n+1)(3p+2n-2).$$

If n' denote the number in the longest row, then $n'=p+n-1$, and therefore $p=n'-n+1$; and the formula may be written

$$s=\frac{n}{6}(n+1)(3n'-n+1),$$

in which n denotes the number in the width, and n' in the length, of the bottom course.

When the pile is incomplete, compute the number in the pile as if complete, then the number in that part of the pile which is lacking, and take the difference of the results.

Exercise 127.

1. Find the fiftieth term of 1, 3, 8, 20, 43, ...

2. Find the sum of the series 4, 12, 29, 55, ... to 20 terms.

3. Find the twelfth term of 4, 11, 28, 55, 92, ...

4. Find the sum of the series 43, 27, 14, 4, -3, ... to 12 terms.

5. Find the seventh term of 1, 1.235, 1.471, 1.708, ...

6. Find the sum of the series 70, 66, 62.3, 58.9, ... to 15 terms.

7. Find the eleventh term of 343, 337, 326, 310, ...

8. Find the sum of the series 7×13, 6×11, 5×9, ... to 9 terms.

9. Find the sum of n terms of the series 3×8, 6×11, 9×14, 12×17, ...

10. Find the sum of n terms of the series 1, 6, 15, 28, 45, ...

11. Determine the number of shot in the side of the base of a triangular pile which contains 286 shot.

12. The number of shot in the top course of a square pile is 169, and in the bottom course 1089. How many shot are there in the pile?

13. Find the number of shot in a rectangular pile having 17 shot in one side of the base and 42 in the other.

14. Find the number of shot in five courses of an incomplete triangular pile which has 15 in one side of the base.

15. The number of shot in a triangular pile is to the number in a square pile, of the same number of courses, as 22 : 41. Find the number of shot in each pile.

16. Find the number of shot required to complete a rectangular pile having 15 and 6 shot, respectively, in the sides of its top course.

17. How many shot must there be in the lowest course of a triangular pile so that 10 courses of the pile, beginning at the base, may contain 37,020 shot?

18. Find the number of shot in a complete rectangular pile of 15 courses which has 20 shot in the longest side of its base.

19. Find the number of shot in the bottom row of a square pile which contains 2600 more shot than a triangular pile of the same number of courses.

20. Find the number of shot in a complete square pile in which the number of shot in the base and the number in the fifth course above differ by 225.

21. Find the number of shot in a rectangular pile which has 600 in the lowest course and 11 in the top row.

Interpolation.

444. As the expansion of $(a+b)^n$ has the same form for fractional as for integral values of n, the formula

$$a_{n+1} = a + nb + \frac{n(n-1)}{1 \times 2}c + \frac{n(n-1)(n-2)}{1 \times 2 \times 3}d + \cdots\cdots$$

may be extended to cases in which n is a fraction, and be used to *interpolate* terms in a series between given terms.

(1) The cube roots of 27, 28, 29, 30, are 3, 3.03659, 3.07232, 3.10723. Find the cube root of 27.9.

	3	3.03659	3.07232	3.10723
First differences,	0.03659	0.03573	0.03491	
Second differences,		-0.00086	-0.00082	
Third differences,		0.00004		

These values substituted in the general formula give

$$3+\frac{9}{10}(0.03659)+\frac{9}{10}\left(-\frac{1}{10}\right)\left(-\frac{0.00086}{2}\right)+\frac{9}{10}\left(-\frac{1}{10}\right)\left(-\frac{11}{10}\left(\frac{0.00004}{6}\right)\right)$$

$$=3+0.032931+0.0000387+0.00000066=3.03297.$$

(2) Given $\log 127 = 2.1038$, $\log 128 = 2.1072$, $\log 129 = 2.1106$. Find $\log 127.37$.

$$\begin{array}{lll} 0.0034 & 0.0034 & = \text{first order of differences.} \\ & 0 & = \text{second order of differences.} \end{array}$$

Therefore, the differences of the second order will vanish, and the required logarithm will be

$$\begin{aligned} & 2.1038 + \tfrac{37}{100} \text{ of } 0.0034 \\ & = 2.1038 + 0.001258 \\ & = 2.1051. \end{aligned}$$

(3) The latitude of the moon on a certain Monday at noon was $1° 53' 18.9''$, at midnight $2° 27' 8.6''$; on Tuesday at noon $2° 58' 55.2''$, at midnight $3° 28' 5.8''$; on Wednesday at noon $3° 54' 8.8''$. Find its latitude at 9 P.M. on Monday.

The series expressed in seconds, and the differences, will be

$$\begin{array}{ccccccccc} 6798.9 & & 8828.6 & & 10735.2 & & 12485.8 & & 14048.8 \\ & 2029.7 & & 1906.6 & & 1750.6 & & 1563.0 & \\ & & -123.1 & & -156.0 & & -187.6 & & \\ & & & -32.9 & & -31.6 & & & \\ & & & & 1.3 & & & & \end{array}$$

As 9 hours $= \frac{3}{4}$ of 12 hours, $n = \frac{3}{4}$.

Also, $a = 6798.9$, $b = 2029.7$, $c = -123.1$, $d = -32.9$, $e = 1.3$.

These values substituted in the general formula

$$a + nb + \frac{n(n-1)}{1 \times 2} c + \frac{n(n-1)(n-2)}{1 \times 2 \times 3} d$$

$$+ \frac{n(n-1)(n-2)(n-3)e}{1 \times 2 \times 3 \times 4} + \cdots\cdots$$

give $6798.9 + \frac{3}{4}(2029.7) + \frac{3}{4}\left(-\frac{1}{4}\right)\left(-\frac{123.1}{2}\right) + \frac{3}{4}\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)\left(-\frac{32.9}{6}\right)$

$$+ \frac{3}{4}\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)\left(-\frac{9}{4}\right)\left(\frac{1.3}{24}\right) + \cdots\cdots$$

$$= 6798.9 + 1522.27 + 11.54 - 1.29 - 0.03 \cdots\cdots$$

$$= 8331.4 = 2° \ 18' \ 51.4''.$$

Compound Series.

445. It is evident from the form of certain series that they are the sum or the difference of two other series.

(1) Find the sum of the series

$$\frac{1}{1\times 2},\ \frac{1}{2\times 3},\ \frac{1}{3\times 4},\ \cdots\cdots,\ \frac{1}{n(n+1)}.$$

Each term of this series may evidently be expressed in two parts

$$\frac{1}{1}-\frac{1}{2},\ \frac{1}{2}-\frac{1}{3},\ \cdots\cdots,\ \frac{1}{n}-\frac{1}{n+1};$$

so that the sum will be

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right),$$

in which the second part of each term, except the last, is cancelled by the first part of the next succeeding term.

Hence, the sum is $1-\dfrac{1}{n+1}$.

As n increases without limit, this sum approaches 1 as a limit.

(2) Find the sum of the series

$$\frac{1}{3\times 5},\ \frac{1}{4\times 6},\ \frac{1}{5\times 7},\ \cdots\cdots,\ \frac{1}{n(n+2)}.$$

Each term may be written,

$$\frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right),\ \frac{1}{2}\left(\frac{1}{4}-\frac{1}{6}\right),\ \cdots\cdots,\ \frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+2}\right).$$

$$\therefore \text{ Sum} = \frac{1}{2}\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots\cdots+\frac{1}{n}-\frac{1}{5}-\frac{1}{6}-\cdots\cdots\right.$$
$$\left.-\frac{1}{n}-\frac{1}{n+1}-\frac{1}{n+2}\right)=\frac{1}{2}\left(\frac{1}{3}+\frac{1}{4}-\frac{1}{n+1}-\frac{1}{n+2}\right).$$

Hence, the sum is $\dfrac{7}{24}-\dfrac{1}{2(n+1)}-\dfrac{1}{2(n+2)}$.

As n increases without limit, this sum approaches $\frac{7}{24}$ as a limit.

Exercise 128.

Sum to n terms, and to infinity, the following series:

1. $\frac{1}{1\times 4}, \frac{1}{2\times 5}, \frac{1}{3\times 6}, \cdots\cdots$

2. $\frac{1}{1\times 3\times 5}, \frac{1}{2\times 4\times 6}, \frac{1}{3\times 5\times 7}, \cdots\cdots$

3. $\frac{1}{2\times 4\times 6}, \frac{1}{4\times 6\times 8}, \frac{1}{6\times 8\times 10}, \cdots\cdots$

4. $\frac{4}{2\times 3\times 4}, \frac{7}{3\times 4\times 5}, \frac{10}{4\times 5\times 6}, \cdots\cdots$

5. $\frac{1}{1\times 2\times 3}, \frac{1}{2\times 3\times 4}, \frac{1}{3\times 4\times 5}, \cdots\cdots$

446. Reversion of a Series. Given

$$y = ax + bx^2 + cx^3 + dx^4 + \cdots\cdots$$

where the series is convergent, to find x in terms of y.

Assume $\quad x = Ay + By^2 + Cy^3 + Dy^4 + \cdots\cdots$

In this series for y put $ax + bx^2 + cx^3 + dx^4 + \cdots\cdots$; then

$$x = aAx + \left.\begin{matrix} bA \\ +a^2B \end{matrix}\right| x^2 + \left.\begin{matrix} cA \\ +2abB \\ +a^3C \end{matrix}\right| x^3 + \cdots\cdots$$

Comparing coefficients (§ 330),

$$aA = 1;\ bA + a^2B = 0;\ cA + 2abB + a^3C = 0.$$

$$\therefore A = \frac{1}{a},\ B = -\frac{b}{a^3},\ C = \frac{2b^2 - ac}{a^5}, \text{ etc.}$$

(1) Given $y = x + x^2 + x^3 + \cdots\cdots$; find x in terms of y.

Here, $a = 1, \quad b = 1, \quad c = 1, \quad d = 1, \ \ldots\ldots$

$A = 1, \quad B = -1, \quad C = 1, \quad D = -1, \ \ldots\ldots$

Hence, $x = y - y^2 + y^3 - y^4 + \cdots\cdots$

(2) Revert $y = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots\cdots$

Here, $a = 1, \quad b = -\frac{1}{2}, \quad c = \frac{1}{3}, \quad d = -\frac{1}{4}, \ \ldots\ldots$

$\therefore \ A = 1, \quad B = \dfrac{1}{\lfloor 2}, \quad C = \dfrac{1}{\lfloor 3}, \quad D = \dfrac{1}{\lfloor 4}, \ \ldots\ldots$

Hence, $x = y + \dfrac{y^2}{\lfloor 2} + \dfrac{y^3}{\lfloor 3} + \dfrac{y^4}{\lfloor 4} + \cdots\cdots$

Exercise 129.

Revert:

1. $y = x - 2x^2 + 3x^3 - 4x^4 + \cdots\cdots$

2. $y = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots\cdots$

3. $y = x + \dfrac{x^2}{1 \cdot 2} + \dfrac{x^3}{2 \cdot 3} + \dfrac{x^4}{3 \cdot 4} + \cdots\cdots$

447. Recurring Series. From the expression $\dfrac{1 + x}{1 - 2x - x^2}$ we obtain by actual division the infinite series

$$1 + 3x + 7x^2 + 17x^3 + 41x^4 + 99x^5 + \cdots\cdots$$

In this series any required term after the second is found by multiplying the term before the required term by $2x$, the term before that by x^2, and adding the products.

Thus, take the fifth term:

$$41x^4 = 2x(17x^3) + x^2(7x^2).$$

In general, if u_n represent the *nth* term,

$$u_n = 2xu_{n-1} + x^2u_{n-2}.$$

A series in which a relation of this character exists is called a **recurring series.** Recurring series are of the *first, second, third, ... order*, according as each term is dependent upon *one, two, three*, ... preceding terms.

A recurring series of the first order is evidently an ordinary geometrical series.

In an arithmetical, or geometrical, series any required term can be found when the term immediately preceding is given. In a series of differences, or a recurring series, several preceding terms must be given if any required term is to be found.

The relation which exists between the successive terms is called the **identical relation** of the series; the coefficients of this relation, when all the terms are transposed to the left member, is called the **scale of relation** of the series.

Thus, in the series

$$1 + 3x + 7x^2 + 17x^3 + 41x^4 + 99x^5 + \dots\dots$$

the identical relation is

$$u_n = 2xu_{n-1} + x^2u_{n-2};$$

and the scale of relation is

$$1 - 2x - x^2.$$

448. If the identical relation of the series is given, any required term can be found when a sufficient number of preceding terms are given.

Conversely, the identical relation can be found when a sufficient number of terms are given.

(1) Find the identical relation of the recurring series

$$1 + 4x + 14x^2 + 49x^3 + 171x^4 + 597x^5 + 2084x^6 + \cdots$$

Try first a relation of the second order.

Assume $$u_n = pxu_{n-1} + qx^2u_{n-2}.$$

Putting $n = 3$, and, then, $n = 4$,

$$14 = 4p + q,$$
$$49 = 14p + 4q;$$

whence, $$p = \tfrac{7}{2},\quad q = 0.$$

This gives a relation which does not hold true for the fifth and following terms.

Try next a relation of the third order.

Assume $$u_n = pxu_{n-1} + qx^2u_{n-2} + rx^3u_{n-3}.$$

Putting $n = 4$, then $n = 5$, then $n = 6$.

$$49 = 14p + 4q + r,$$
$$171 = 49p + 14q + 4r,$$
$$597 = 171p + 49q + 14r;$$

whence, $$p = 3,\quad q = 2,\quad r = -1.$$

This gives the relation

$$u_n = 3xu_{n-1} + 2x^2u_{n-2} - x^3u_{n-3}$$

which is found to hold true for the seventh term.

The *scale of relation* is $1 - 3x - 2x^2 + x^3$.

(2) Find the eighth term of the above series.

Here, $$\begin{aligned} u_8 &= 3xu_7 + 2x^2u_6 - x^3u_5 \\ &= 3x(2084x^6) + 2x^2(597x^5) - x^3(171x^4) \\ &= 7275x^7. \end{aligned}$$

449. Sum of an Infinite Series. By the *sum* of an infinite convergent *numerical* series is meant the limit which the sum of n terms of the series approaches as n is indefinitely increased; a *divergent* numerical series has no true sum.

By the sum of an infinite series of which the successive terms involve one or more *variables* is meant the *generating function* of the series, that is, *the expression of which the series is the expansion.*

The generating function is a true sum when, and only when, the series is convergent.

The process of finding the generating function is called *summation* of the series.

450. Sum of a Recurring Series. The sum of a recurring series can be found by a method analogous to that by which the sum of a geometrical series is found (§ 314).

Take, for example, a recurring series of the second order in which the identical relation is

$$u_k = pu_{k-1} + qu_{k-2},$$

or

$$u_k - pu_{k-1} - qu_{k-2} = 0.$$

Let s represent the sum of the series; then

$$\begin{aligned} s &= u_1 + u_2 + u_3 + \cdots \quad u_{n-1} + u_n, \\ -ps &= \quad -pu_1 - pu_2 - \cdots - pu_{n-2} - pu_{n-1} - pu_n, \\ -qs &= \qquad -qu_1 - \cdots - qu_{n-3} - qu_{n-2} - qu_{n-1} - qu_n. \end{aligned}$$

Now, by the identical relation,

$$u_3 - pu_2 - qu_1 = 0,\ u_4 - pu_3 - qu_2 = 0, \cdots u_n - pu_{n-1} - qu_{n-2} = 0.$$

Therefore, adding the above series,

$$s = \frac{u_1 + (u_2 - pu_1)}{1 - p - q} - \frac{pu_n + q(u_n + u_{n-1})}{1 - p - q}.$$

Observe that the denominator is the *scale of relation.*

If the series is infinite and convergent, u_n and u_{n-1} each approaches 0 as a limit, and s approaches as a limit the fraction $\dfrac{u_1 + (u_2 - pu_1)}{1 - p - q}$.

If the series is infinite, whether convergent or not, this fraction is the *generating function* of the series.

For a recurring series of the third order of which the identical relation is

$$u_k = pu_{k-1} + qu_{k-2} + ru_{k-3},$$

we find
$$s = \frac{u_1 + (u_2 - pu_1) + (u_3 - pu_2 - qu_1)}{1 - p - q - r}$$
$$- \frac{pu_n + q(u_n + u_{n-1}) + r(u_n + u_{n-1} + u_{n-2})}{1 - p - q - r}.$$

Similarly for any recurring series.

(1) Find the generating function of the infinite recurring series

$$1 + 4x + 13x^2 + 43x^3 + 142x^4 + \cdots\cdots$$

By § 448 the identical relation is found to be

$$u_k = 3xu_{k-1} + x^2u_{k-2}.$$

Hence,
$$\begin{aligned} s &= 1 + 4x + 13x^2 + 43x^3 + 142x^4 + \cdots\cdots \\ -3xs &= \quad - 3x - 12x^2 - 39x^3 - 129x^4 - \cdots\cdots \\ -x^2s &= \qquad\quad - x^2 - 4x^3 - 13x^4 - \cdots\cdots \end{aligned}$$

Adding,
$$(1 - 3x - x^2)s = 1 + x,$$
$$s = \frac{1 + x}{1 - 3x - x^2}.$$

(2) Find the generating function and the general term of the infinite recurring series

$$1 - 7x - x^2 - 43x^3 - 49x^4 - 307x^5 - \cdots\cdots$$

Here
$$u_k = xu_{k-1} + 6x^2u_{k-2}.$$

$$\begin{aligned} s &= 1 - 7x - x^2 - 43x^3 - 49x^4 - \cdots\cdots \\ -xs &= \quad - x + 7x^2 + x^3 + 43x^4 + \cdots\cdots \\ -6x^2s &= \qquad\quad - 6x^2 + 42x^3 + 6x^4 + \cdots\cdots \end{aligned}$$

$$s = \frac{1 - 8x}{1 - x - 6x^2} = \frac{1 - 8x}{(1 + 2x)(1 - 3x)}.$$

By § 331 we find

$$\frac{1-8x}{(1+2x)(1-3x)}=\frac{2}{1+2x}-\frac{1}{1-3x}.$$

By the binomial theorem or by actual division,

$$\frac{1}{1+2x}=1-2x+2^2x^2-2^3x^3+\cdots\cdots+2^r(-1)^rx^r+\cdots\cdots,$$

$$\frac{1}{1-3x}=1+3x+3^2x^2+3^3x^3+\cdots\cdots 3^rx^r+\cdots\cdots$$

Hence the general term of the given series is

$$[2^{r+1}(-1)^r-3^r]x^r.$$

(3) Find the identical relation in the series

$$1^2+2^2+3^2+4^2+5^2+6^2+7^2+\cdots\cdots$$

The identical relation is found from the equations

$$16=9p+4q+r,$$
$$25=16p+9q+4r,$$
$$36=25p+16q+9r,$$

to be

$$u_k=3u_{k-1}-3u_{k-2}+u_{k-3}.$$

Exercise 130.

Find the identical relation and generating function of:

1. $1+2x+7x^2+23x^3+76x^4+\cdots\cdots$
2. $3+2x+3x^2+7x^3+18x^4+\cdots\cdots$

Find the generating function and general term of:

3. $2+3x+5x^2+9x^3+17x^4+33x^5+\cdots\cdots$
4. $7-6x+9x^2+27x^3+54x^4+189x^5+\cdots\cdots$
5. $1+5x+9x^2+13x^3+17x^4+21x^5+\cdots\cdots$
6. $1+x-7x^3+33x^4-130x^5+499x^6+\cdots\cdots$
7. $3+6x+14x^2+36x^3+98x^4+276x^5+\cdots\cdots$

Find the sum of n terms of:

8. $2+5+10+17+26+37+50+\cdots\cdots$
9. $1^3+2^3+3^3+4^3+5^3+\cdots\cdots$

Exponential and Logarithmic Series.

451. Exponential Series. By the binomial theorem

$$\left(1+\frac{1}{n}\right)^{nx}=1+nx\times\frac{1}{n}+\frac{nx(nx-1)}{1\times 2}\times\frac{1}{n^2}$$

$$+\frac{nx(nx-1)(nx-2)}{1\times 2\times 3}\times\frac{1}{n^3}+\cdots\cdots$$

$$=1+x+\frac{x\left(x-\frac{1}{n}\right)}{\lfloor 2}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\lfloor 3}+\cdots\cdots \qquad (1)$$

This equation is true for all real values of x, but is only true for values of n numerically greater than 1, since $\frac{1}{n}$ must be numerically less than 1 (§ 438).

As (1) is true for all values of x, it is true when $x=1$.

$$\therefore\ \left(1+\frac{1}{n}\right)^{n}=1+1+\frac{1-\frac{1}{n}}{\lfloor 2}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\lfloor 3}+\cdots\cdots \qquad (2)$$

But $$\left[\left(1+\frac{1}{n}\right)^{n}\right]^{x}=\left(1+\frac{1}{n}\right)^{nx}.$$

Hence from (1) and (2),

$$\left[1+1+\frac{1-\frac{1}{n}}{\lfloor 2}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\lfloor 3}+\cdots\cdots\right]^{x}$$

$$=1+x+\frac{x\left(x-\frac{1}{n}\right)}{\lfloor 2}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\lfloor 3}+\cdots\cdots$$

This last equation is true for all values of n numerically greater than 1. Take the limits of the two members as n increases without limit. Then (§ 427)

$$\left(1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\cdots\cdots\right)^x=1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\cdots\cdots, \qquad (3)$$

and this is true for all values of x. It is easily seen by § 436 that the second series is convergent for all values of x; the first series was proved convergent in § 435.

The sum of the infinite series in parenthesis is called the base of the natural system of logarithms, and is generally represented by e; hence, by (3),

$$e^x=1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\cdots\cdots \qquad \textbf{A}$$

To calculate the value of e we proceed as follows:

	1.000000
2	1.000000
3	0.500000
4	0.166667
5	0.041667
6	0.008333
7	0.001388
8	0.000198
9	0.000025
	0.000003

Adding, $e=2.71828+$

452. In **A** put cx in place of x; then

$$e^{cx}=1+cx+\frac{c^2x^2}{\lfloor 2}+\frac{c^3x^3}{\lfloor 3}+\cdots\cdots$$

Put $e^c=a$; then $c=\log_e a$, and $e^{cx}=a^x$.

$$\therefore\ a^x=1+\log_e a+\frac{x^2(\log_e a)^2}{\lfloor 2}+\frac{x^3(\log_e a)^3}{\lfloor 3}+\cdots\cdots \qquad \textbf{B}$$

The series in **B** is known as the **exponential series**; **B** reduces to **A** when we put e for a.

453. Logarithmic Series. In **A** put $e^x = 1 + y$; then

$$x = \log_e(1+y), \text{ and by } \mathbf{A},$$

$$y = x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \cdots\cdots$$

Revert the series (§ 446), and we obtain

$$x = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots\cdots$$

But $\qquad x = \log_e(1+y).$

$$\therefore \log_e(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots\cdots \qquad \mathbf{C}$$

Similarly from **B**,

$$\log_a(1+y) = \frac{1}{\log_e a}\left(y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots\cdots\right). \qquad \mathbf{D}$$

The series in **D** is known as the **logarithmic series**; **D** reduces to **C** when we put e for a.

In **C** and **D** y must be between -1 and $+1$, or be equal to $+1$, in order to have the series convergent (§ 439, Ex. 1).

454. Modulus. Comparing **C** and **D** we obtain

$$\log_a(1+y) = \frac{1}{\log_e a}\log_e(1+y);$$

or, putting N for $1+y$,

$$\log_a N = \frac{1}{\log_e a}\log_e N.$$

Hence, to change logarithms from the base e to the base a, multiply by $\dfrac{1}{\log_e a} = \log_a e$; and conversely.

The number by which *natural logarithms* must be multiplied to obtain logarithms to the base a is called the **modulus** of the system of logarithms of which a is the base.

Thus, the modulus of the common system is $\log_{10} e$.

455. Calculation of Logarithms. Since the series in **C** and **D** are not convergent when x is numerically greater than 1, they are not adapted to the calculation of logarithms in general. We obtain a convenient series as follows:

The equation

$$\log_e(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots\cdots \tag{1}$$

holds true for all values of y numerically less than 1; therefore, if it holds true for any particular value of y, it will hold true when we put $-y$ for y; this gives

$$\log_e(1-y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \cdots\cdots \tag{2}$$

Subtracting (2) from (1), since

$$\log_e(1+y) - \log_e(1-y) = \log_e\left(\frac{1+y}{1-y}\right),$$

we find $\log_e\left(\frac{1+y}{1-y}\right) = 2\left(y + \frac{y^3}{3} + \frac{y^5}{5} + \cdots\cdots\right).$

Put $y = \frac{1}{2z+1}$; then $\frac{1+y}{1-y} = \frac{z+1}{z}$,

and
$$\log_e\left(\frac{z+1}{z}\right) = \log_e(z+1) - \log_e z$$

$$= 2\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \cdots\cdots\right). \qquad \mathbf{E}$$

This series is convergent for all positive values of z.

Logarithms to any base a can be calculated by the corresponding series obtained from **D**; viz.:

$$\log_a(z+1) - \log_a z$$
$$= \frac{2}{\log_e a}\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \cdots\cdots\right). \qquad \mathbf{F}$$

(1) Calculate to six places of decimals $\log_e 2$, $\log_e 3$, $\log_e 10$, $\log_{10} e$.

In **E** put $\quad z = 1$; then $2z + 1 = 3$, $\log_e z = 0$,

and
$$\log_e 2 = \frac{2}{3} + \frac{2}{3 \times 3^3} + \frac{2}{5 \times 3^5} + \frac{2}{7 \times 3^7} + \cdots\cdots$$

The work may be arranged as follows:

$$\begin{array}{r|l}
3 & 2.0000000 \\
9 & 0.6666667 \div 1 = 0.6666667 \\
9 & 0.0740741 \div 3 = 0.0246914 \\
9 & 0.0082305 \div 5 = 0.0016461 \\
9 & 0.0009145 \div 7 = 0.0001306 \\
9 & 0.0001016 \div 9 = 0.0000113 \\
9 & 0.0000113 \div 11 = 0.0000010 \\
 & 0.0000013 \div 13 = \underline{0.0000001} \\
 & \qquad\qquad \log_e 2 = 0.693147+
\end{array}$$

$$\log_e 3 = \log_e 2 + \frac{2}{5} + \frac{2}{3 \times 5^3} + \frac{2}{5 \times 5^5} + \cdots\cdots$$
$$= 1.0986123.$$
$$\log_e 9 = \log_e (3^2) = 2 \log_e 3 = 2.1972246.$$
$$\log_e 10 = \log_e 9 + \frac{2}{19} + \frac{2}{3 \times 19^3} + \frac{2}{5 \times 19^5} + \cdots\cdots$$
$$= 2.1972246 + 0.1053606$$
$$= 2.302585.$$
$$\log_{10} e = \frac{1}{\log_e 10} = 0.434294.$$

Hence, the *modulus* of the common system is 0.434294.

To *ten* places of decimals:

$$\log_e 10 = 2.3025850928,$$
$$\log_{10} e = 0.4342944819.$$

For calculating common logarithms we use the series in **F**.

$$\log_{10}(z+1) - \log_{10} z$$
$$= 0.8685889638\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \cdots\cdots\right).$$

(2) Calculate to five places of decimals $\log_{10} 11$.

Put $z = 10$; then $2z + 1 = 21$, $\log z = 1$.

$$\log 11 = 1 + 0.868588\left(\frac{1}{21} + \frac{1}{3 \times 21^3} + \frac{1}{5 \times 21^5} + \cdots\right)$$

$$\begin{array}{r|l}
21 & 0.868588 \\
441 & 0.041361 \div 1 = 0.041361 \\
 & \quad\;\; 94 \div 3 = 31 \\
 & \hline \quad\;\; 0.041392 \\
 & \quad\;\; 1 \\
\end{array}$$

$$\log_{10} 11 = 1.04139$$

In calculating logarithms, the accuracy of the work may be tested every time we come to a composite number by adding together the logarithms of the several factors (§ 347). In fact the logarithms of composite numbers may be found by addition, and then only the logarithms of prime numbers need be found by the series.

456. Limit of $\left(1 + \frac{x}{n}\right)^n$**.** By the binomial theorem,

$$\left(1+\frac{x}{n}\right)^n = 1 + n \times \frac{x}{n} + \frac{n(n-1)}{1 \times 2} \times \frac{x^2}{n^2} + \frac{n(n-1)(n-2)}{1 \times 2 \times 3} \times \frac{x^3}{n^3} + \cdots$$

$$= 1 + x + \frac{1 - \frac{1}{n}}{\lfloor 2} x^2 + \frac{\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)}{\lfloor 3} x^3 + \cdots$$

This equation is true for all values of n greater than x (§ 438). Take the limit as n increases without limit, x remaining finite; then

$$\lim_{n \text{ infinite}} \left(1 + \frac{x}{n}\right)^n = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots$$

$$= e^x$$

$$= \lim_{n \text{ infinite}} \left(1 + \frac{1}{n}\right)^{nx}. \qquad \S\, 451$$

Exercise 131.

Determine whether the following infinite series are convergent or divergent:

1. $1+\frac{1}{2}+\frac{1}{3^2}+\frac{1}{4^3}+\cdots\cdots$

2. $1+\frac{1^2}{\lfloor 2}+\frac{2^3}{\lfloor 3}+\frac{3^4}{\lfloor 4}+\cdots\cdots$

3. $\frac{2}{1^2}+\frac{3}{2^2}+\frac{4}{3^2}+\frac{5}{4^2}+\cdots\cdots$

4. $\frac{1}{1^m}+\frac{1^m}{2^m}+\frac{2^m}{3^m}+\frac{3^m}{4^m}+\cdots\cdots$

5. Show that the infinite series

$$\frac{1}{1\times 2}-\frac{1}{2\times 2^2}+\frac{1}{3\times 2^3}-\frac{1}{4\times 2^4}+\cdots\cdots$$

is convergent, and find its sum.

6. Find the limit which $\sqrt[n]{1+nx}$ approaches as n approaches 0 as a limit.

7. Prove that $\frac{1}{e}=2\left(\frac{1}{\lfloor 3}+\frac{2}{\lfloor 5}+\frac{3}{\lfloor 7}+\cdots\cdots\right)$.

8. Calculate to four places, $\log_e 4$, $\log_e 5$, $\log_e 6$, $\log_e 7$.

9. Find to four places the moduli of the systems of which the bases are: 2, 3, 4, 5, 6, 7.

10. Show that

$$\log_e\left(\frac{8}{e}\right)=\frac{5}{1\times 2\times 3}+\frac{7}{3\times 4\times 5}+\frac{9}{5\times 6\times 7}+\cdots\cdots$$

11. Show that

$$\log_e a-\log_e b=\frac{a-b}{a}+\frac{1}{2}\left(\frac{a-b}{a}\right)^2+\frac{1}{3}\left(\frac{a-b}{a}\right)^3+\cdots\cdots$$

12. Show that, if x is positive,

$$x+\frac{1}{x}-\frac{1}{2}\left(x^2+\frac{1}{x^2}\right)+\frac{1}{3}\left(x^3+\frac{1}{x^3}\right)-\cdots\cdots=\log_e\left(2+x+\frac{1}{x}\right).$$

13. Show that $1+\frac{2^3}{\lfloor 2}+\frac{3^3}{\lfloor 3}+\frac{4^3}{\lfloor 4}=5e$.

CHAPTER XXXV.

GENERAL PROPERTIES OF EQUATIONS.

457. Functions. Any expression involving x is called a **function** of x. If x is involved only in powers and roots, the expression is an **algebraic** function of x. A **rational** algebraic function involves x only in powers, not in roots. A rational algebraic function of x is **integral** if it involves only *positive* integral powers of x.

Thus, $3x^2$, $\sqrt{x^2-1}$, $ax^{\frac{2}{3}}+b$, $\sqrt{\dfrac{a+x}{a-x}}$, $\dfrac{(ax^2+bx+c)^{\frac{1}{3}}}{dx+e}$, a^x, $\log(x)$ are functions of x, the first five being algebraic. $\dfrac{x^2-1}{x^2+1}$, $\dfrac{x^3+3x+5}{2x^2+3}$, $\dfrac{ax^3+bx^2+cx+d}{px^5+qx^3+r}$, are fractional rational functions of x. $3x^2+4x-1$, ax^3+bx+c, cx^3-d, are integral rational functions of x.

458. Hereafter only integral rational functions, called **quantics**, will be considered unless it is otherwise expressly stated. The **degree** of such a function is the same as the exponent of the highest power of x involved.

For brevity a function of x is often represented by $f(x)$, $F(x)$, $\phi(x)$, or by some similar notation. The value of the function $f(x)$ when we put a for x is represented by $f(a)$.

Thus, if $\qquad f(x)=2x^3-3x^2-4x+5,$

$$f(2)=2(2)^3-3(2)^2-4(2)+5=1.$$

459. Equations. An equation which involves only integral rational functions of x is called a **rational integral**

equation. Every such equation can be reduced, by transposing all the terms to the first member, to the **general form**

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots\cdots + a_n = 0,$$

or briefly, $$f(x) = 0.$$

The **degree** of the equation is the same as that of the function $f(x)$. An equation of the first degree is called a **linear** equation. Those of higher degrees are called in order **quadratic, cubic, biquadratic, quintic,** etc.

The **roots** of an equation are those values of x for which the function $f(x)$ vanishes.

460. Fundamental Theorems. Theorem I. *If the function* $\mathbf{f(x)}$ *is divisible by* $\mathbf{x - h}$, *then* $\mathbf{h}$ *is a root of the equation* $\mathbf{f(x) = 0}$. For, if $\phi(x)$ is the quotient obtained by dividing $f(x)$ by $x - h$, we have

$$f(x) = (x - h)\,\phi(x),$$

and the equation $f(x) = 0$ may be written

$$(x - h)\,\phi(x) = 0.$$

But h is obviously a root of this equation.

461. Theorem II. *Conversely, if* $\mathbf{h}$ *is a root of the equation* $\mathbf{f(x) = 0}$, *then* $\mathbf{f(x)}$ *is divisible by* $\mathbf{x - h}$.

For example, consider the equation

$$f(x) = ax^3 + bx^2 + cx + d = 0.$$

Now, since h is a root of the equation $f(x) = 0$, we have

$$0 = ah^3 + bh^2 + ch + d.$$

Subtracting,

$$f(x) = a(x^3 - h^3) + b(x^2 - h^2) + c(x - h).$$

But every term of the second number is divisible by $x - h$, and consequently $f(x)$ is also divisible by $x - h$. Similarly for any other equation.

462. Synthetic Division. Let the function

$$2x^5 - 3x^4 - 5x^3 + x^2 - 33x - 7$$

be divided by $x - 3$.

The work is as follows:

$$\begin{array}{l|l}
2x^5 - 3x^4 - 5x^3 + \quad x^2 - 33x - 7 & x - 3 \\
\underline{2x^5 - 6x^4} & \overline{2x^4 + 3x^3 + 4x^2 + 13x + 6} \\
\qquad 3x^4 - 5x^3 & \\
\qquad \underline{3x^4 - 9x^3} & \\
\qquad\qquad 4x^3 + \quad x^2 & \\
\qquad\qquad \underline{4x^3 - 12x^2} & \\
\qquad\qquad\qquad 13x^2 - 33x & \\
\qquad\qquad\qquad \underline{13x^2 - 39x} & \\
\qquad\qquad\qquad\qquad 6x - 7 & \\
\qquad\qquad\qquad\qquad \underline{6x - 18} & \\
\qquad\qquad\qquad\qquad\qquad 11 &
\end{array}$$

The work may be abridged by omitting the powers of x, and writing only the coefficients. We then have

$$\begin{array}{l|l}
2 - 3 - 5 + 1 - 33 - 7 & 1 - 3 \\
\underline{2 - 6} & \overline{2 + 3 + 4 + 13 + 6} \\
\quad 3 - 5 & \\
\quad \underline{3 - 9} & \\
\qquad 4 + 1 & \\
\qquad \underline{4 - 12} & \\
\qquad\quad 13 - 33 & \\
\qquad\quad \underline{13 - 39} & \\
\qquad\qquad 6 - 7 & \\
\qquad\qquad \underline{6 - 18} & \\
\qquad\qquad\quad 11 &
\end{array}$$

But the operation may be still further abridged. As the first term of the divisor is unity, the first term of each remainder is the next term of the quotient. Again, we need not bring down the several terms of the dividend. Finally, we need not write the first terms of the partial products.

The operation is then as follows:

$$\begin{array}{rrrrrrl}
2 & -3 & -5 & +1 & -33 & -7 & \underline{|1-3} \\
 & -6 & & & & & \\ \hline
 & 3 & & & & & \\
 & & -9 & & & & \\ \hline
 & & 4 & & & & \\
 & & & -12 & & & \\ \hline
 & & & 13 & & & \\
 & & & & -39 & & \\ \hline
 & & & & 6 & & \\
 & & & & & -18 & \\ \hline
 & & & & & 11 &
\end{array}$$

Omitting the first term of the divisor as superfluous, changing -3 to $+3$, and adding instead of subtracting, we have, on raising the terms and bringing down the first coefficient,

$$\begin{array}{rrrrrrl}
2 & -3 & -5 & +\ \ 1 & -33 & -\ \ 7 & \underline{|3} \\
 & 6 & +9 & +12 & +39 & +18 & \\ \hline
2 & +3 & +4 & +13 & +\ \ 6 & +11 &
\end{array}$$

The last term below the line gives us the remainder, the preceding terms the coefficients of the quotient. In this particular problem the quotient is $2x^4 + 3x^3 + 4x^2 + 13x + 6$, and the remainder is 11.

This method is called the method of **Synthetic Division.** For the application of this method to the division of any quantic by $x - h$ we have the following rule:

Write the coefficients **a**, **b**, **c**, *etc., in a horizontal line.*

Bring down the first coefficient **a**.

Multiply **a** *by* **h**, *and add the product to* **b**.

Multiply the sum so obtained by **h**, *and add the product to* **c**.

Continuing this process, the last sum will be the remainder, and the preceding sums the coefficients of the quotient.

Remark. If there are any powers of x missing, their places are to be supplied by zero coefficients.

463. Value of a Quantic. By the principles of division it is evident that the operation of dividing a given quantic $f(x)$ by $x-h$ can be carried on until the remainder does not involve x. Represent the quotient by $\phi(x)$, and the remainder by R. Then we have

$$f(x)=(x-h)\phi(x)+R.$$

Putting h for x,

$$f(h)=0+R.$$

Hence the value which a quantic $\mathbf{f(x)}$ *assumes when we put* $\mathbf{h}$ *for* $\mathbf{x}$ *is equal to the last remainder obtained in the operation of dividing* $\mathbf{f(x)}$ *by* $\mathbf{x-h}$.

Exercise 132.

Find the quotient and remainder obtained by dividing each of the following expressions by the divisor opposite it.

1.	x^3+5x^2-7x-3	$x-2.$
2.	$x^4-7x^3+5x^2-10x+12$	$x-3.$
3.	$2x^4+3x^3-6x^2-4x-24$	$x-2.$
4.	$x^5-3x^3+2x^2-5$	$x-4.$
5.	$3x^4-6x^2+7x-10$	$x+3.$
6.	x^3+3x^2+x+4	$x+\sqrt{-1}.$

Are the following numbers roots of the equations opposite them?

7.	(2)	$x^3-3x^2+4x+4=0.$
8.	(-3)	$x^4-3x^3+7x^2-9x+84=0.$
9.	(-5)	$x^5+6x^4+7x^3+9x^2-5=0.$
10.	(0.2)	$x^3-2.2x^2+3.4x-0.6=0.$

Find the values of the following expressions when for x we put the number opposite the expression.

11. $2x^3+3x^2+5x-10$ (2).

12. $3x^5-5x^4+2x^2+3x-15$ (3).

13. $x^4-4x^3+7x^2+9x+12$ (-3).

464. Number of Roots. We shall assume that every rational integral equation has at least one root. The proof of this truth is beyond the scope of the present chapter.*

Let $f(x)=0$ be a rational integral equation of the nth degree. This equation has, by assumption, at least one root. Let α_1 be a root.

Then, by § 460, $f(x)=(x-\alpha_1)f_1(x)$,

where $f_1(x)$ is a quantic of degree $n-1$.

The equation $f_1(x)=0$ must, by assumption, have a root. Let α_2 be a root.

Then, by § 460, $f_1(x)=(x-\alpha_2)f_2(x)$,

where $f_2(x)$ is a quantic of degree $n-2$.

Continuing this process, we see that at each step the degree of the quotient is diminished by one. Hence, we can find n factors $x-\alpha_1$, $x-\alpha_2 \ldots\ldots x-\alpha_n$. The last quotient will not involve x, and is readily seen to be a_0, the coefficient of x^n in $f(x)$.

Now,
$$\begin{aligned} f(x) &= (x-\alpha_1)f_1(x) \\ &= (x-\alpha_1)(x-\alpha_2)f_2x \\ &\cdots\cdots \\ &= a_0(x-\alpha_1)(x-\alpha_2)\ldots\ldots(x-\alpha_n), \end{aligned}$$

---

* See Burnside and Panton, *Theory of Equations*, 2d ed., Art. 195; Briot et Bouquet, *Fonctions Elliptiques*, Art. 23.

so that the equation $f(x)=0$ may be written

$$a_0(x-\alpha_1)(x-\alpha_2)\ldots\ldots(x-\alpha_n)=0,$$

which is evidently satisfied if x has any one of the n values $\alpha_1, \alpha_2 \ldots\ldots \alpha_n$.

It follows, then, that if *every* rational integral equation has one root, *an equation of the nth degree has n roots.*

465. Multiple Roots. The n roots of an equation of the nth degree are not necessarily all different.

Thus the equation $x^3-4x^2-3x+18=0$ may be written $(x+2)(x-3)(x-3)=0$, and its roots are therefore $-2, 3, 3$.

The root 3 and the corresponding factor $x-3$ occur twice; hence 3 is said to be a double root. A root which occurs three times is called a triple root; four times a quadruple root; and so on. Any root which occurs more than once is called a **multiple** root.

466. Solutions by Trial. When all the roots of an equation but two can be found by trial, the equation can be readily solved by the process of § 464. The work can be much abbreviated by employing the method of synthetic division (§ 462).

Solve the equation

$$x^4-x^3-9x^2+11x+6=0.$$

Try $+1$ and -1. Substituting these values for x, we have

$$1-1-9+11+6=0,$$
$$1+1-9-11+6=0.$$

Both these equations are false, so that neither $+1$ nor -1 is a root.

Try 2. Dividing by $x-2$,

$$\begin{array}{l} 1-1-9+11+6\,\underline{|2} \\ \quad\;+2+2-14-6 \\ \hline 1+1-7-\;\;3\quad 0 \end{array}$$

we see that 2 is a root. The quotient is x^3+x^2-7x-3.

In this quotient try 2 again. Dividing by $x-2$,

$$\begin{array}{l|l} 1+1-7-3 & \underline{2} \\ \quad +2+6-2 & \\ \hline 1+3-1-5 & \end{array}$$

we see that 2 is not again a root.

Try -2. Dividing by $x+2$,

$$\begin{array}{l|l} 1+1-7-3 & \underline{-2} \\ \quad -2+2+10 & \\ \hline 1-1-5+7 & \end{array}$$

we see that -2 is not a root.

Try -3. Dividing by $x+3$,

$$\begin{array}{l|l} 1+1-7-3 & \underline{-3} \\ \quad -3+6+3 & \\ \hline 1-2-1+0 & \end{array}$$

we see that -3 is a root. The quotient is x^2-2x-1.

Hence the given equation may be written

$$(x-2)(x+3)(x^2-2x-1)=0.$$

Therefore one of these three factors must vanish.

If $x-2=0$, $x=2$; if $x+3=0$, $x=-3$; if $x^2-2x-1=0$, we find on solving this quadratic that $x=1+\sqrt{2}$, or $x=1-\sqrt{2}$. Hence the four roots of the given equation are

$$2,\ -3,\ 1+\sqrt{2},\ 1-\sqrt{2}.$$

Exercise 133.

Solve the equations:

1. $x^3-3x+2=0.$
2. $x^3+x^2-16x+20=0.$
3. $x^3-8x^2+21x-18=0.$
4. $x^3-x^2-8x+12=0.$
5. $x^3+3x^2-4=0.$

6. $x^4 + 2x^3 - 11x^2 - 12x + 36 = 0.$

7. $x^4 - x^3 - 10x^2 + 4x + 24 = 0.$

8. $x^4 - 4x^3 - 18x^2 + 108x - 135 = 0.$

9. $x^5 - 40x^3 + 160x^2 - 240x + 128 = 0.$

10. $x^5 - 4x^4 + 13x^3 - 52x^2 + 36x - 144 = 0.$

11. $x^4 + 2x^3 - 5x^2 - 12x - 4 = 0.$

467. Roots Given. When all the roots of an equation are given, the equation can at once be written.

Write the equation of which the roots are 1, 2, 4, -5.

The equation is $\quad (x-1)(x-2)(x-4)(x+5) = 0,$

or $\quad x^4 - 2x^3 - 21x^2 + 62x - 40 = 0.$

468. Relations between the Roots and the Coefficients. The quadratic equation of which the roots are α and β is (§ 269)

$$(x-\alpha)(x-\beta) = 0;$$

or, multiplying out,

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0.$$

The cubic equation of which the roots are α, β, γ is

$$(x-\alpha)(x-\beta)(x-\gamma) = 0;$$

or, $\quad x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \alpha\gamma + \beta\gamma)x - \alpha\beta\gamma = 0.$

The biquadratic equation of which the roots are α, β, γ, δ is

$$(x-\alpha)(x-\beta)(x-\gamma)(x-\delta) = 0;$$

or,

$$x^4 - (\alpha + \beta + \gamma + \delta)x^3 + (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)x^2$$
$$- (\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)x + \alpha\beta\gamma\delta = 0.$$

And so on.

Take any equation in which the highest power of x has the coefficient unity. From the above we have the following relations between the roots and the coefficients:

The coefficient of the *second* term, with its sign changed, is equal to the sum of the roots.

The coefficient of the *third* term is equal to the sum of all the products that can be formed by taking the roots *two* at a time.

The coefficient of the *fourth* term, with its sign changed, is equal to the sum of all the products that can be formed by taking the roots *three* at a time.

The coefficient of the *fifth* term is equal to the sum of all the products that can be formed by taking the roots *four* at a time; and so on.

If the number of roots is *even*, the *last* term is equal to the product of all the roots. If the number of roots is *odd*, the *last* term, *with its sign changed*, is equal to the product of all the roots.

Observe that the sign of the coefficient is changed when an odd number of roots are taken to form a product; that the sign is unchanged when an even number of roots are taken to form a product.

469. Imaginary Roots. If an imaginary number is a root of an equation with real coefficients, the conjugate imaginary (§ 237) is also a root.

Let $\alpha + \beta i$, where $i = \sqrt{-1}$, be a root of the equation

$$a_0x^n + a_1x^{n-1} + \cdots\cdots a_n = 0,$$

the coefficients being real.

Put $\alpha + \beta i$ for x in the left member of the equation, and expand the powers of $\alpha + \beta i$ by the binomial theorem. All the terms which do not contain i, and all the terms which contain even powers of i, will be real; all the terms which

contain odd powers of i will be imaginary. Representing the real part of the result by P, and the imaginary part of the result by Qi, we have (§ 459), since $\alpha+\beta i$ is a root,

$$P+Qi=0,$$

and therefore $P=0$ and $Q=0$ (§ 240).

Now put $\alpha-\beta i$ for x in the given equation. The result may be obtained from the former result by changing i to $-i$. The even powers of i will be unchanged while the odd powers will have their signs changed. The real part will therefore be unchanged, and the imaginary part changed only in sign. The result is

$$P-Qi,$$

which vanishes, since by the preceding $P=0$ and $Q=0$.

Therefore $\alpha-\beta i$ is a root of the given equation (§ 459).

This theorem is generally stated as follows: *Imaginary roots enter equations in pairs.*

It follows from this theorem that an equation of odd degree has always at least one real root. Thus an equation of the third degree must have three real roots or one real root and two imaginary roots.

Exercise 134.

Form the equation of which the roots are

1. $2, 3, -5$. **3.** $2, -3, -2$. **5.** $3, 0, -4$.

2. $3, 1, -2$. **4.** $3, 4, -6$. **6.** $6, 2, 3\frac{1}{2}$.

7. $3+\sqrt{2}, 3-\sqrt{2}, -6$. **9.** $1, 3, -2, -4$.

8. $1+\sqrt{3}, 1-\sqrt{3}, \frac{1}{2}$. **10.** $2, \frac{1}{2}, -2, -\frac{1}{2}$.

11. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, -\frac{1}{2}$.

12. $1+\sqrt{2}, 1-\sqrt{2}, \sqrt{3}+1, -\sqrt{3}+1$.

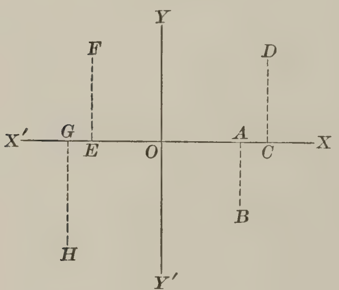
13. $2+\sqrt{-1}, 2-\sqrt{-1}, 1+2\sqrt{-1}, 1-2\sqrt{-1}$.

14. $1, -2, 3, -4, 5$. **15.** $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}, 2, 3$.

Graphical Representation of Functions.

The investigation of the changes in the value of $f(x)$ corresponding to changes in the value of x is much facilitated by using the system of graphical representation explained in the following sections.

470. Co-ordinates. Let $X'X$ and $Y'Y$ be two perpendicular straight lines drawn in a plane, intersecting at O.



The lines $X'X$ and $Y'Y$ are called **axes of reference**; the point O is called the **origin.**

Distances measured from O along $X'X$, as OA, OC, OE, and OG, are called **abscissas**; distances measured from $X'X$ parallel to $Y'Y$, as AB, CD, EF, and GH, and called **ordinates.**

Abscissas are considered positive if measured to the right; negative, if measured to the left. Ordinates are considered positive if measured upwards; negative, if measured downwards.

Thus, OA, OC, CD, and EF are positive; OE, OG, AB, and GH are negative.

An abscissa is generally represented by x; an ordinate is generally represented by y.

The abscissa and ordinate of any point are called the **co-ordinates** of that point. Thus the co-ordinates of B are OA and AB.

The co-ordinates of a point are written thus: (x, y).

Thus, (7, 4) is the point of which the abscissa is 7 and the ordinate 4.

The axis $X'X$ is called the **axis of abscissas**, or the **axis of x**; the axis $Y'Y$, the **axis of ordinates**, or the **axis of y.**

471. It is evident that if a point B is given, its co-ordinates referred to given axes may be found by drawing the ordinate and measuring the distances OA and AB.

Conversely, if the co-ordinates of a point are given, the point may be readily constructed.

Thus, to construct the point $(7, -4)$, a convenient length is taken as a unit of length. A distance of 7 units is laid off on OX to the *right* from O to A. At A a perpendicular to $X'X$ is drawn *downwards*, of length 4 units, to B. Then B is the required point.

Construct the points $(3, 2)$; $(5, 4)$; $(6, -3)$; $(-4, -3)$; $(-4, 2)$; $(-3, -5)$; $(4, -3)$.

472. Graph of a Function. Let $f(x)$ be any function of x, where x is a variable. Put $y = f(x)$; then y is a new variable connected with x by the relation $y = f(x)$. If $f(x)$ is a rational integral function of x, it is evident that to every value of x corresponds one, and only one, value of y.

If different values of x be laid off as abscissas, and the corresponding values of $f(x)$ as ordinates, the points thus obtained will all lie on a line; this line will generally be a curved line, or, as it is briefly called, a *curve*. This curve is called the **graph** of the function $f(x)$; it is also called the **locus** of the equation $y = f(x)$.

We proceed to construct the graphs of several functions.

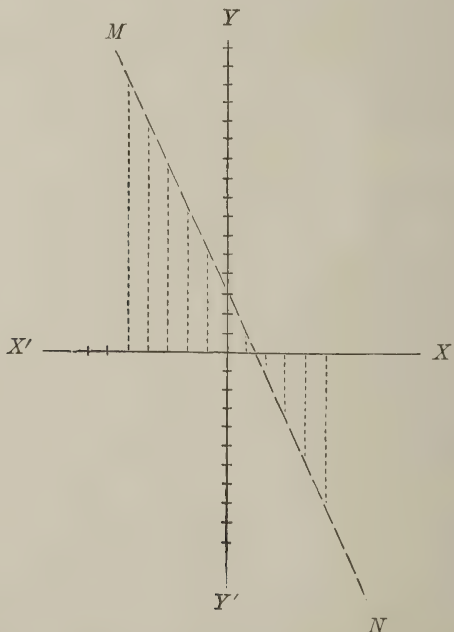
Remark. In constructing, or *plotting*, as it is called, the graph of a function, the student will find it convenient to use the paper called plotting, or co-ordinate, paper. This is ruled in small squares, and therefore saves much labor.

(1) Construct the graph of $3 - 2x$.

Put $y = 3 - 2x$. The following table is readily computed:

If $x = 1$, $y = 1$.	If $x = -1$, $y = 5$.
" $x = 2$, $y = -1$.	" $x = -2$, $y = 7$.
" $x = 3$, $y = -3$.	" $x = -3$, $y = 9$.
" $x = 4$, $y = -5$.	" $x = -4$, $y = 11$.
" $x = 5$, $y = -7$.	" $x = -5$, $y = 13$.

Constructing the above points, it appears that the graph of the function $3 - 2x$ is the straight line MN.



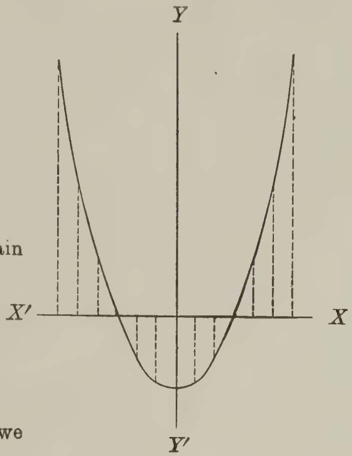
In general, when the equation $y = f(x)$ contains only the first powers of x and y, the graph will be a straight line.

(2) Plot the graph of $\frac{1}{2}x^2 - 4$.

Putting $y = \frac{1}{2}x^2 - 4$, we readily compute the following table:

If	$x = 0$,	$y = -4$.
"	$x = \pm 1$,	$y = -3.5$.
"	$x = \pm 2$,	$y = -2$.
"	$x = \pm 3$,	$y = +0.5$.
"	$x = \pm 4$,	$y = +4$.
"	$x = \pm 5$,	$y = +8.5$.
"	$x = \pm 6$,	$y = +14$.

Plotting these points, we obtain the curve here given.

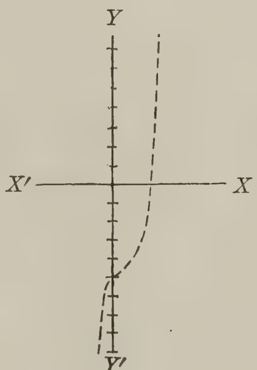


(3) Plot the graph of

$$x^3 - x^2 + x - 5.$$

Putting $y = x^3 - x^2 + x - 5$, we compute the following table:

If x is	y is
0.5,	-4.625.
1.0,	-4.000.
1.5,	-2.375.
2.0,	$+1.000$.
2.5,	$+6.875$.
0.0,	-5.000.
-0.5,	-5.875.
-1.5,	-12.125.

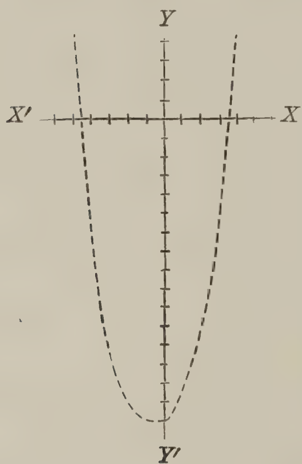


Interpolation shows that if $y = 0$, $x = 1.88+$. Does the result agree with the figure?

473. Consider any rational integral function of x, for example, $x^2 + x - \frac{63}{4}$.

Put $$y = x^2 + x - \frac{63}{4}.$$

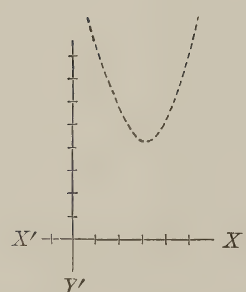
Assuming values of x, we compute the corresponding values of y, and construct the graph. Now, any value of x which makes $y=0$ satisfies the equation $x^2+x-\frac{63}{4}=0$, and is a root of that equation; hence, any abscissa whose corresponding ordinate is zero represents a root of this equation. The roots may be found, approximately, by measuring the abscissas of the points where the graph meets XX', for at these points $y=0$.



From the given equation the following table may be formed:

If x is	y is	If x is	y is
0,	-15.75.	-1,	-15.75.
1,	-13.75.	-2,	-13.75.
2,	$-\ 9.75$.	-3,	$-\ 9.75$.
3,	$-\ 3.75$.	-4,	$-\ 3.75$.
4,	$+\ 4.25$.	-5,	$+\ 4.25$.

The table shows that one root is between 3 and 4 (since y changes from $-$ to $+$, and therefore passes through zero); and, for a like reason, the other is between -4 and -5.



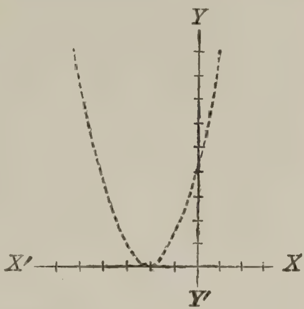
474. An equation of any degree may be thus plotted, and the graph will be found to cross the axis $X'X$ as many times as there are *real* roots in the equation.

When an equation has no real root, the graph does not meet $X'X$.

In the equation $x^2-6x+13=0$, both of whose roots are imaginary, the graph, at its nearest approach, is 4 units distant from $X'X$.

If an equation has a double root, its graph touches $X'X$, but does not intersect it.

The equation $x^2 + 4x + 4 = 0$ has the roots -2 and -2, and the graph is as shown in the figure.



Exercise 135.

Construct the graphs of the following functions:

1. $x^2 + 3x - 10$.
2. $x^3 - 2x^2 + 1$.
3. $x^4 - 20x^2 + 64$.
4. $x^2 - 4x + 10$.
5. $x^4 - 5x^2 + 4$.
6. $x^3 - 4x^2 + x - 1$.

Derivatives.

475. Increments. If $f(x)$ is any function of x, then, if x is a variable, $f(x)$ will also be a variable. If we assign to x any particular value a, $f(x)$ will take the particular value $f(a)$. If we increase x to $a + h$, $f(x)$ will take the new value $f(a + h)$. The increase, $f(a + h) - f(a)$, in the value of $f(x)$, is called the **increment** of $f(x)$ between $x = a$ and $x = a + h$. In general the increase, $f(x + h) - f(x)$, in the value of $f(x)$ between any initial value, x, of the variable and a final value $x + h$ is called the increment of $f(x)$ between these two values of x.

Thus, if $f(x) = x^3$, the increment of $f(x)$ from $x = 2$ to $x = 3$ is $3^3 - 2^3 = 27 - 8 = 19$. This increment is not equally distributed over the interval from $x = 2$ to $x = 3$. Thus, if we divide the interval into three equal parts $x = 2$ to $x = 2\frac{1}{3}$, $x = 2\frac{1}{3}$ to $x = 2\frac{2}{3}$, $x = 2\frac{2}{3}$ to $x = 3$, the increments of x^3 for these parts are

$$(2\tfrac{1}{3})^3 - 2^3 = \tfrac{127}{27},\ (2\tfrac{2}{3})^3 - (2\tfrac{1}{3})^3 = \tfrac{169}{27},\ \text{and}\ 3^3 - (2\tfrac{2}{3})^3 = \tfrac{217}{27}.$$

In general, for equal increments of x, the increments of $f(x)$ will be unequal; that is, as x varies, $f(x)$ changes its values at a varying *rate*.

476. Derivatives. The ratio, $\dfrac{f(x+h)-f(x)}{h}$, of the increment of $f(x)$ to the corresponding increment of x is *the average rate of change* of $f(x)$ in the interval from x to $x+h$.

The *limit* of this ratio, as h approaches 0, is called the **derivative** of $f(x)$. The derivative of $f(x)$ with respect to x is in general a new function of x and is denoted by $D_x f(x)$ or by $f'(x)$.

We have then

$$f'(x) = D_x f(x) = \underset{h \doteq 0}{\text{limit}}\left[\frac{f(x+h)-f(x)}{h}\right].$$

NOTE. $h \doteq 0$ is read "as h approaches 0."

The rule for finding the derivative of a function is therefore as follows:

In the given function change **x** *to* **x + h.**

From the new value of the function subtract the old, and divide the remainder by **h.**

Take the limit of the quotient as **h** *approaches zero as a limit.*

The derivative of a constant is 0, since the increment of a constant will always be 0.

Find $D_x(x^3)$.

The function is	x^3.
Change x to $x+h$,	$(x+h)^3$.
From the new value subtract the old,	$(x+h)^3 - x^3$,
or	$3hx^2 + 3h^2x + h^3$.
Divide by h,	$3x^2 + 3hx + h^2$.

Take the limit as h approaches 0 as a limit, and we have

$$3x^2.$$

$$\therefore\ D_x(x^3) = 3x^2.$$

477. Derivative of x^n. The function is x^n. Changing x to $x+h$, we obtain $(x+h)^n$. Now, whatever the value of n, $(x+h)^n$ can be expanded by the binomial theorem, and we obtain

$$(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{\lfloor 2}x^{n-2}h^2 + \cdots\cdots$$

From this new value of the function subtract x^n, the old value, and divide by h.

We now have

$$D_x(x^n) = \underset{h \doteq 0}{\text{limit}}\left[nx^{n-1} + \frac{n(n-1)}{\lfloor 2}x^{n-2}h + \cdots\cdots\right]$$

$$= nx^{n-1};$$

the sum of the terms after the first approaches 0 as a limit by § 438.

Hence, to find the derivative with respect to x of any power of x, *multiply by the exponent, and diminish the exponent of x by one.*

Thus, $\quad D_x(x^4) = 4x^3;\quad D_x(x^{-3}) = -3x^{-4};$

$$D_x\frac{1}{\sqrt{x^3}} = D_x(x^{-\frac{3}{2}}) = -\tfrac{3}{2}x^{-\frac{5}{2}}.$$

(1) Find the derivative of ax^n.

We have, $\quad D_x(ax^n) = \underset{h \doteq 0}{\text{limit}}\left[\frac{a(x+h)^n - ax^n}{h}\right]$

$$= a \times \underset{h \doteq 0}{\text{limit}}\left[\frac{(x+h)^n - x^n}{h}\right].$$

But this, as we have just seen, $= anx^{n-1}$.

In general, if a function is multiplied by a constant, its derivative is multiplied by the same constant.

(2) Find the derivative of

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots\cdots + a_n.$$

We have

$D_x f(x)$

$$= \underset{h \doteq 0}{\text{limit}}\left[\frac{a_0(x+h)^n + a_1(x+h)^{n-1} + \cdots\cdots + a_n - a_0x^n - a_1x^{n-1} - \cdots\cdots - a_n}{h}\right]$$

$$= \underset{h \doteq 0}{\text{limit}}\left[\frac{a_0(x+h)^n - a_0x^n + a_1(x+h)^{n-1} - a_1x^{n-1} + \cdots\cdots + a_{n-1}(x+h) - a_{n-1}}{h}\right]$$

$$= \underset{h \doteq 0}{\text{limit}}\left[\frac{a_0(x+h)^n - a_0x^n}{h}\right] + \underset{h \doteq 0}{\text{limit}}\left[\frac{a_1(x+h)^{n-1} - a_1x^{n-1}}{h}\right] + \cdots\cdots$$

$$= D_x(a_0x^n) + D_x(a_1x^{n-1}) + \cdots\cdots$$

$$= na_0x^{n-1} + (n-1)a_1x^{n-2} + \cdots\cdots + a_{n-1}.$$

That is, to differentiate any rational integral function, we differentiate each term separately, and add the results together.

Exercise 136.

Find the derivatives of the following functions:

1. x^2. **2.** x^3. **3.** x^4. **4.** $3x^5$. **5.** x^{-2}. **6.** $\dfrac{2}{x^3}$.

7. x^3+2x. **8.** x^2+3x+4. **9.** $3x^4+2x^3+5x^2+6x+4$.

10. $(x+3)^2$. **11.** $(x+a)^3$. **12.** $(x+1)^{-2}$. **13.** $\dfrac{1}{x^2-1}$.

478. Derivative of a Product. Suppose that the product consists of three factors,

$$f(x) = (x-a)(x-b)(x-c).$$

We have

$D_x f(x)$

$$= \underset{h = 0}{\text{limit}}\left[\frac{(x+h-a)(x+h-b)(x+h-c)-(x-a)(x-b)(x-c)}{h}\right],$$

or, if we denote $x-a$ by α, $x-b$ by β, $x-c$ by γ,

$$D_x f(x)$$

$$=\lim_{h \doteq 0}\left[\frac{(\alpha+h)(\beta+h)(\gamma+h)-\alpha\beta\gamma}{h}\right]$$

$$=\lim_{h \doteq 0}\left[\frac{\alpha\beta\gamma+h(\alpha\beta+\beta\gamma+\gamma\alpha)+h^2(\alpha+\beta+\gamma)+h^3-\alpha\beta\gamma}{h}\right]$$

$$=\lim_{h \doteq 0}\left[\frac{h(\alpha\beta+\beta\gamma+\gamma\alpha)+h^2(\alpha+\beta+\gamma)+h^3}{h}\right]$$

$$=\lim_{h \doteq 0}[\alpha\beta+\beta\gamma+\gamma\alpha+h(\alpha+\beta+\gamma)+h^2]$$

$$=\alpha\beta+\beta\gamma+\gamma\alpha$$

$$=(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a).$$

Similarly if $f(x)$ consist of four factors,

$$f(x)=(x-a)(x-b)(x-c)(x-d),$$

$$f'(x)=D_x f(x)=(x-b)(x-c)(x-d)+(x-c)(x-d)(x-a)$$
$$+(x-d)(x-a)(x-b)+(x-a)(x-b)(x-c),$$

and in general we have the result:

The derivative of any product of n *factors*

$$(x-a)(x-b)(x-c)(x-d)\cdots\cdots$$

is the sum of n *terms, each of which consists of the product of* n − 1 *of the same factors.*

479. Multiple Factors. Suppose that

$$f(x)=(x-a)(x-b)(x-c)$$

as before, but let $b=a$. We have then from the preceding section

$$f'x=D_x f(x)=(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)$$
$$=(x-a)^2+(x-a)(x-c)+(x-c)(x-a),$$

and $x-a$ is a factor of $f'(x)$ as well as of $f(x)$.

Similarly in the case of four factors,

$$f(x) = (x-a)(x-b)(x-c)(x-d),$$

every term of $f'(x)$ contains either $x-a$ or $x-b$ as a factor. Consequently when $b=a$, $f'(x)$ will contain $x-a$ as a factor in every term.

In general, we have the proposition

If $\mathrm{f}(\mathbf{x})$ *contain a double factor* $(\mathbf{x}-\mathbf{a})^2$, $\mathrm{f}'(\mathbf{x})$ *will contain the same factor as a simple factor.*

By the same reasoning it is shown that if $f(x)$ contain a triple factor $(x-a)^3$, $f'(x)$ will contain $(x-a)^2$ as a factor, and in general

If $\mathrm{f}(\mathbf{x})$ *contain a factor* $(\mathbf{x}-\mathbf{a})^k$, $\mathrm{f}'(\mathbf{x})$ *will contain as a factor* $(\mathbf{x}-\mathbf{a})^{k-1}$.

480. Multiple Roots. Let the equation $f(x)=0$ have a multiple root a of order k. Then $f(x)$ contains $(x-a)^k$ as a factor, and consequently $f'(x)$ contains $(x-a)^{k-1}$ as a factor. The H. C. F. of $f(x)$ and $f'(x)$ therefore contains $(x-a)^{k-1}$ as a factor.

To find the multiple roots of the equation $f(x)=0$.

Find the H. C. F. of $f(x)$ and $f'(x)$ and resolve it into its factors. Each multiple root will occur once more in the equation $f(x)=0$ than the corresponding factor occurs in the H. C. F.

Find the multiple roots of the equation

$$x^5 - 15x^3 - 10x^2 + 60x + 72 = 0.$$

Here $f(x) = x^5 - 15x^3 - 10x^2 + 60x + 72,$

$$f'(x) = 5x^4 - 45x^2 - 20x + 60.$$

Find the H. C. F. of $f(x)$ and $f'(x)$, as follows:

$$\begin{array}{r|r|l}
5)\underline{5+0-45-20+60} & 1+0-15-10+60+72 & 1 \\
1+0-\ \ 9-\ \ 4+12 & \underline{1+0-\ \ 9-\ \ 4+12} & \\
\underline{1+1-\ \ 8-12} & -6)\underline{-\ \ 6-\ \ 6+48+72} & \\
-1-\ \ 1+\ \ 8+12 & 1+\ \ 1-\ \ 8-12 & 1-1 \\
\underline{-1-\ \ 1+\ \ 8+12} & &
\end{array}$$

$\therefore$ The H. C. F. is $x^3 + x^2 - 8x - 12$.

We find, by trial, that 3 is a root of the equation

$$x^3 + x^2 - 8x - 12 = 0.$$

The other roots are found to be -2 and -2.

Hence $$x^3 + x^2 - 8x - 12 = (x-3)(x+2)^2.$$

Therefore 3 is a double root and -2 a triple root of the given equation. As the equation is of the fifth degree, these are all the roots, and the equation may be written

$$(x-3)^2(x+2)^3 = 0.$$

Having found the multiple roots of an equation, we may divide by the corresponding factors, and find the remaining roots, if any, from the reduced equation.

Exercise 137.

The following equations have multiple roots. Find all the roots of each equation.

1. $x^3 - 5x^2 + 7x - 3 = 0.$
2. $x^3 - 3x^2 + 4 = 0.$
3. $x^4 - 2x^3 - 7x^2 + 20x - 12 = 0.$
4. $x^4 - 2x^3 - 11x^2 + 12x + 36 = 0.$
5. $x^4 - 24x^2 + 64x - 48 = 0.$
6. $x^5 + x^4 - 17x^3 - 21x^2 + 72x + 108 = 0.$
7. $x^6 - 5x^5 + 5x^4 + 9x^3 - 14x^2 - 4x + 8 = 0.$

Transformation of Equations.

481. The solution of an equation, and the investigation of its properties, is often facilitated by a change in the form of the equation. Such a change of form is called a **transformation** of the equation.

482. Roots with Signs changed. *The roots of the equation* $f(-x) = 0$ *are those of the equation* $f(x) = 0$, *each with its sign changed.*

For, let α be any root of equation $f(x) = 0$.

Then, we must have $f(\alpha) = 0$.

In the quantic $f(-x)$ put $-\alpha$ for x; that is, α for $-x$.

The result is $f(\alpha)$.

But we have just seen that $f(\alpha)$ vanishes, since α is a root of the equation $f(x) = 0$. Hence, $f(-x)$ vanishes when we put $-\alpha$ for x, and (§ 459) $-\alpha$ is therefore a root of the equation $f(-x) = 0$.

To obtain the $f(-x)$ we change the sign of all the *odd* powers of x in the quantic $f(x)$.

Thus, the roots of the equation

$$x^4 - 2x^3 - 13x^2 + 14x + 24 = 0$$

are 2, 4, -1, -3; and those of the equation

$$x^4 + 2x^3 - 13x^2 - 14x + 24 = 0$$

are -2, -4, $+1$, $+3$.

483. Roots multiplied by a Given Number. Consider the equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0. \qquad (1)$$

Put $y = mx$, then $x = \dfrac{y}{m}$; and the equation becomes

$$a\left(\frac{y}{m}\right)^4 + b\left(\frac{y}{m}\right)^3 + c\left(\frac{y}{m}\right)^2 + d\left(\frac{y}{m}\right) + e = 0. \qquad (2)$$

Let α be any root of (1); the left-member of (1) vanishes when we put α for x, and we obtain

$$a\alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha + e = 0.$$

In the left-member of (2), put $m\alpha$ for y; we obtain

$$a\alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha + e,$$

which, as we have just seen, vanishes. Hence, if α is a root of (1), $m\alpha$ is a root of (2).

Similarly, for an equation of any degree.

Equation (2) may be written in the form

$$ay^4 + mby^3 + m^2cy^2 + m^3dy + m^4e = 0.$$

The above form, if written with x in place of y, gives the following rule:

Multiply the second term by m; the third term by m^2; and so on. Zero coefficients are to be supplied for missing powers of x.

Write the equation of which the roots are the doubles of the roots of the equation

$$3x^4 - 2x^3 + 4x^2 - 6x - 5 = 0.$$

Here $m = 2$, and the result is

$$3x^4 - 2(2)x^3 + 4(2)^2x^2 - 6(2)^3x - 5(2)^4 = 0,$$

or $$3x^4 - 4x^3 + 16x^2 - 48x - 80 = 0.$$

484. Removal of Fractional Coefficients. If any of the coefficients of an equation in the form

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots\cdots p_n = 0$$

are fractions, we can remove fractions as follows:

Multiply the roots by m; then take m so that all of the coefficients will be integers.

NOTE. Every equation can be reduced to this form, called the p form, by dividing through by the coefficient of x^n.

Reduce to an equation, in the p form, with integral coefficients,

$$2x^3 - \tfrac{1}{3}x^2 + \tfrac{5}{6}x + \tfrac{1}{4} = 0.$$

Dividing by 2, $\quad x^3 - \tfrac{1}{6}x^2 + \tfrac{5}{12}x + \tfrac{1}{8} = 0.$

Multiplying the roots by m (§ 483),

$$x^3 - \frac{m}{6}x^2 + \frac{5m^2}{12}x + \frac{m^3}{8} = 0.$$

The least value of m that will render the coefficients all integral is seen to be 6. Putting 6 for m, we obtain

$$x^3 - x^2 + 15x + 27 = 0,$$

the equation required.

Any multiple of 6 might have been used instead of 6, but the smaller the number, the easier the work.

485. Reciprocal Equations. Consider the equations

$$ax^4 + bx^3 + cx^2 + dx + e = 0, \tag{1}$$

$$ex^4 + dx^3 + cx^2 + bx + a = 0. \tag{2}$$

The coefficients in the one equation are the same as those in the other reversed in order.

If α be any root of (1), then $\frac{1}{\alpha}$ will be a root of (2). For if α be a root of (1), we have

$$a\alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha + e = 0; \tag{3}$$

and if $\frac{1}{\alpha}$ be a root of (2), we have

$$\frac{e}{\alpha^4} + \frac{d}{\alpha^3} + \frac{c}{\alpha^2} + \frac{b}{\alpha} + a = 0,$$

or $$e + d\alpha + c\alpha^2 + b\alpha^3 + a\alpha^4 = 0. \tag{4}$$

But the equations (3) and (4) are identically the same.

The four roots of equation (2) are therefore the reciprocals of the four roots of equation (1).

Similarly for equations of any degree.

The coefficients of an equation may be such that reversing their order does not change the equation. In this case

the reciprocal of a root is another root of the equation. That is, one half the roots are reciprocals of the other half.

An equation in which this is true is called a **reciprocal equation.**

(1) Write the equation of which the roots are the reciprocals of the roots of

$$2x^5 + 4x^4 - 3x^3 + 7x^2 + 2x - 5 = 0.$$

The result is

$$-5x^5 + 2x^4 + 7x^3 - 3x^2 + 4x + 2 = 0,$$

or $$5x^5 - 2x^4 - 7x^3 + 3x^2 - 4x - 2 = 0.$$

(2) The equation $6x^5 - 29x^4 + 27x^3 + 27x^2 - 29x + 6 = 0$ in a reciprocal equation.

Its roots are found to be -1, 2, 3, $\frac{1}{2}$, $\frac{1}{3}$. Here -1 is the reciprocal of itself; $\frac{1}{2}$ is the reciprocal of 2; and $\frac{1}{3}$ of 3.

486. Roots diminished by a Given Number. Consider the equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0. \tag{1}$$

To obtain the equation which has for its roots the roots of the above equation each diminished by h, we proceed as follows:

Put $y = x - h$; then $x = y + h$; and the equation becomes

$$a(y+h)^4 + b(y+h)^3 + c(y+h)^2 + d(y+h) + e = 0. \tag{2}$$

Let α be any root of (1); then we must have

$$a\alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha + e = 0.$$

In the left-member of (2) put α for $y + h$; that is, $\alpha - h$ for y; we obtain

$$a\alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha + e;$$

which, as we have just seen, vanishes.

Hence, $a-h$ is a root of (2). Since the above is true for each of the roots of (1), and the two equations are evidently of the same degree, the roots thus found are all the roots of equation (2).

Similarly for an equation of any degree.

487. Calculation of the Coefficients. The labor of calculating the coefficients in the transformed equation of (§ 486) can be greatly reduced by Synthetic Division.

Suppose equation (2) of (§ 486) to be expanded and arranged in order of the powers of y, and suppose the result to be

$$Ay^4 + By^3 + Cy^2 + Dy + E = 0.$$

Since $y = x - h$, this equation is equivalent to

$$A(x-h)^4 + B(x-h)^3 + C(x-h)^2 + D(x-h) + E = 0. \quad (3)$$

If we divide the left-hand member of (3) by $x-h$, the remainder is E, the last coefficient. And if we divide the resulting quotient again by $x-h$, the remainder is D, the next to the last coefficient, and so on.

Now the left-hand member of (3) is identically the same as the left-hand member of (1). Consequently we have the following rule for transforming any equation $f(x) = 0$ into an equation whose roots are less by h.

Divide $f(x)$ by $x-h$, and the remainder will be the last coefficient in the transformed equation. Divide again by $x-h$, and the remainder will be the coefficient of the last term but one in the transformed equation. Continue the process until all the coefficients are determined.

Obtain the equation which has for its roots the roots of the equation

$$3x^4 - 7x^3 - 4x^2 - 6x + 5 = 0,$$

each diminished by 3.

The work is as follows:

$$\begin{array}{lllll}
3 - & 7 - & 4 - & 6 + & 5 \,\lfloor 3 \\
 & + 9 + & 6 + & 6 + & 0 \\
\hline
3 + & 2 + & 2 + & 0 + & 5 \\
 & + 9 + & 33 + & 105 & \\
\hline
3 + & 11 + & 35 + & 105 & \\
 & 9 + & 60 & & \\
\hline
3 + & 20 + & 95 & & \\
 & 9 & & & \\
\hline
3 + & 29 & & &
\end{array}$$

(5 = first remainder.)

(105 = second remainder.)

(95 = third remainder.)

(29 = fourth remainder.)

The required equation is then

$$3x^4 + 29x^3 + 95x^2 + 105x + 5 = 0.$$

Exercise 138.

Write the equations whose roots are the products of the roots of the following equations by the number opposite:

1. $x^3 - 5x^2 + 2x - 3 = 0$ $\quad (-1)$.

2. $x^4 + 4x^2 + 3x + 5 = 0$ $\quad (+2)$.

3. $2x^3 - 3x^2 + 5x - 7 = 0$ $\quad (-2)$.

4. $5x^4 - 3x^3 + 2x - 6 = 0$ $\quad (+5)$.

Write the equations which have for their roots the reciprocals of the roots of the following equations:

5. $2x^3 + 3x^2 - x - 2 = 0$.

6. $3x^4 + 5x^3 - x^2 + 2x + 3 = 0$.

7. $2x^5 + 3x^4 - 6x^3 + 6x^2 - 3x - 2 = 0$.

Write the equations whose roots are the roots of the following equations diminished by the number opposite:

8. $x^3 + 4x^2 - 12x - 17 = 0$ $\quad (+2)$.

9. $3x^4 - 10x^3 + 2x^2 - 5x + 6 = 0$ $\quad (+3)$.

10. $x^5 - 5x^3 - 4x^2 + 8x + 10 = 0$ $\quad (-2)$.

488. Descartes' Rule of Signs. An equation in which all the powers of x from x^0 to x^n are present is said to be **complete**; if any powers of x are missing, the equation is said to be **incomplete**. An incomplete equation can be made complete by writing the missing powers of x with zero coefficients.

A **permanence** of sign occurs when $+$ follows $+$, or $-$ follows $-$; a **variation** of sign when $-$ follows $+$, or $+$ follows $-$.

Thus, in the complete equation

$$x^6 - 3x^5 + 2x^4 + x^3 - 2x^2 - x - 3,$$

writing only the signs

$$+ \quad - \quad + \quad + \quad - \quad - \quad -,$$

we see that there are three variations of sign and three permanences.

For *positive* roots, Descartes' rule is as follows:

The number of positive roots of the equation $\mathbf{f}(\mathbf{x}) = 0$ *cannot exceed the number of variations of sign in the quantic* $\mathbf{f}(\mathbf{x})$.

To prove this it is only necessary to prove that for every positive root introduced into an equation there is one variation of sign added.

Suppose the signs of a quantic to be

$$+ \quad - \quad + \quad + \quad + \quad - \quad - \quad +$$

and introduce a new positive root. We multiply by $x - h$; or, writing only the signs, by $+\,-$. The result is

$$\begin{array}{ccccccccc}
+ & - & + & + & + & - & - & + & \\
+ & - & & & & & & & \\
\hline
+ & - & + & + & + & - & - & + & \\
 & - & + & - & - & - & + & + & - \\
\hline
+ & - & + & \pm & \pm & - & \mp & + & -
\end{array}$$

The ambiguous signs $\pm$, $\mp$ indicate that there is doubt whether the term is positive or negative. Examining the product we see that to permanences in the multiplicand correspond ambiguities in the product. Hence, we cannot have a greater number of permanences in the product than in the multiplicand, and may have a less number. But there is one more term in the product than in the multiplicand. Hence we have *at least* one more *variation* in the product than in the multiplicand.

For each positive root introduced we have at least one more variation of sign. Hence the number of positive roots cannot exceed the number of variations of sign.

Negative Roots. Change x to $-x$. The negative roots of the given equation will be positive roots of this latter equation (§ 482), and the preceding rule may then be applied.

489. From Descartes' rule we obtain the following:

If the signs of the terms of an equation are all positive, the equation has no positive root.

If the signs of the terms of a complete equation are alternately positive and negative, the equation has no negative root.

If the roots of a complete equation are all real, the number of positive roots is the same as the number of variations of sign, and the number of negative roots is the same as the number of permanences of sign.

490. Existence of Imaginary Roots. In an incomplete equation Descartes' rule sometimes enables us to detect the presence of imaginary roots.

Thus, the equation $$x^3 + 5x + 7 = 0$$

may be written $$x^3 \pm 0x^2 + 5x + 7 = 0.$$

We are at liberty to assume that the second term is positive, or that it is negative.

Taking it positive, we have the signs

$$+ \quad + \quad + \quad +;$$

there is no variation, and the equation has no positive root.

Taking it negative, we have the signs

$$+ \quad - \quad + \quad +;$$

there is but one permanence, and therefore not more than one negative root.

As there are three roots, and as imaginary roots enter in pairs, the given equation has one real negative root and two imaginary roots.

Exercise 139.

All the roots of the equations given below are real; determine their signs.

1. $x^4 + 4x^3 - 43x^2 - 58x + 240 = 0$.
2. $x^3 - 22x^2 + 155x - 350 = 0$.
3. $x^4 + 4x^3 - 35x^2 - 78x + 360 = 0$.
4. $x^3 - 12x^2 - 43x - 30 = 0$.
5. $x^5 - 3x^4 - 5x^3 + 15x^2 + 4x - 12 = 0$.
6. $x^3 - 12x^2 + 47x - 60 = 0$.
7. Show that $x^6 - 3x^2 - x + 1 = 0$ has at least two imaginary roots.
8. Show that $x^4 + 15x^2 + 7x - 11 = 0$ has two imaginary roots, and determine the signs of the real roots.
9. Show that $x^n - 1 = 0$ has but two real roots, $+1$ and -1, when n is even; and but one real root, $+1$, when n is odd.
10. Show that $x^n + 1 = 0$ has no real root when n is even; and but one real root, -1, when n is odd.

CHAPTER XXXVI.

NUMERICAL EQUATIONS.

491. A real root of a numerical equation is either **commensurable** or **incommensurable.**

Commensurable roots are either integers or fractions. Repeating decimals can be expressed as fractions, and roots in that form are consequently commensurable.

Incommensurable roots cannot be found exactly, but may be calculated to any desired degree of accuracy by the method of approximation explained in this chapter.

Commensurable Roots.

492. Integral Roots. The process of finding integral roots given in § 466 is long and tedious when there are many numbers to be tried. The number of divisors to be tried is diminished by the following theorem:

Every integral root of an equation with integral coefficients is a divisor of the last term.

We shall prove this for an equation of the fourth degree, but the proof is perfectly general.

Let h be an integral root of the equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0,$$

where the coefficients a, b, c, d, e are all integers.

Since h is a root,

$$ah^4 + bh^3 + ch^2 + dh + e = 0, \qquad (\S\ 459)$$

or,

$$e = -dh - ch^2 - bh^3 - ah^4.$$

Dividing by h,

$$\frac{e}{h} = -d - ch - bh^2 - ah^3.$$

Since the right member is an integer, the left member must be an integer. That is, e is divisible by h.

Similarly, for any equation with integral coefficients.

Hence, in applying the method of § 466, we need try only divisors of the last term. The necessary labor may be still further reduced by the method shown in the following example.

Find the integral roots of the equation

$$2x^4 - x^3 - 29x^2 + 34x + 24 = 0.$$

Neither $+1$ nor -1 is a root.

The other divisors of 24 are ± 2, ± 3, ± 4, ± 6, ± 8, ± 12, ± 24.

Try $+2$:

$$\begin{array}{l} 2 - 1 - 29 + 34 + 24 \,\underline{|2} \\ \quad\; + 4 + \;\; 6 - 46 - 24 \\ \hline 2 + 3 - 23 - 12 + \;\; 0 \end{array}$$

Hence $+2$ is a root. Dividing the equation by $x - 2$, we have for the resulting equation

$$2x^3 + 3x^2 - 23x - 12 = 0.$$

Try $+3$:

$$\begin{array}{l} 2 + 3 - 23 - 12 \,\underline{|3} \\ \qquad\; 6 + 27 + 12 \\ \hline 2 + 9 + \;\; 4 + \;\; 0 \end{array}$$

Hence 3 is a root.

The remaining roots are roots of the quadratic equation

$$2x^2 + 9x + 4 = 0,$$

and are -4 and $-\frac{1}{2}$.

Therefore the roots of the given equation are 2, 3, -4, $-\frac{1}{2}$.

493. Fractional Roots. *An equation with integral coefficients, in which the coefficient of the highest power of* **x** *is unity, cannot have a rational fraction for a root.*

The general form of such an equation is

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0,$$

where $p_1, p_2, \ldots p_n$ are integers.

If possible let $\frac{h}{k}$, where h and k are integers, and $\frac{h}{k}$ is in its lowest terms, be a root. Then,

$$\frac{h^n}{k^n} + p_1\frac{h^{n-1}}{k^{n-1}} + p_2\frac{h^{n-2}}{k^{n-2}} + \cdots p_n = 0.$$

Multiplying by k^{n-1} and transposing,

$$\frac{h^n}{k} = -p_1h^{n-1} - p_2h^{n-2}k - \cdots - p_nk^{n-1}.$$

Now the right member is an integer; the left member is a fraction in its lowest terms, since h^n and k have no common divisor as h and k have no common divisor, § 415, V. But a fraction in its lowest terms cannot be equal to an integer. Hence $\frac{h}{k}$, or any other rational fraction, cannot be a root.

The real roots of an equation with integral coefficients in the p form are, therefore, integral or incommensurable.

In case an equation has fractional roots, we can find them as follows:

Transform the equation into an equation with integral coefficients by multiplying the roots by some number m (§ 484). Find the integral roots of the transformed equation, and divide each by m.

Solve the equation

$$36x^4 - 55x^2 - 35x - 6 = 0.$$

Write this $$x^4 - \tfrac{55}{36}x^2 - \tfrac{35}{36}x - \tfrac{1}{6} = 0.$$

Multiplying the roots by 6, we obtain

$$x^4 - 55x^2 - 210x - 216 = 0,$$

of which the roots are found to be $-2, -3, -4, 9$.

Hence, the roots of the given equation are

$$-\tfrac{2}{6}, -\tfrac{3}{6}, -\tfrac{4}{6}, \tfrac{9}{6}; \text{ or, } -\tfrac{1}{3}, -\tfrac{1}{2}, -\tfrac{2}{3}, \tfrac{3}{2}.$$

Exercise 140.

Find the commensurable roots and, if possible, all the roots of the following equations:

1. $x^3 + 2x^2 - 40x + 64 = 0.$
2. $x^3 - 3x^2 - 10x + 24 = 0.$
3. $x^4 + x^3 - 36x^2 - 24x - 72 = 0.$
4. $x^4 - 7x^3 - 6x^2 - 18x + 16 = 0.$
5. $x^4 - 9x^3 + 17x^2 + 27x - 60 = 0.$
6. $18x^4 - 33x^3 - 13x^2 + 12x + 4 = 0.$
7. $27x^4 - 72x^3 + 33x^2 + 8x - 4 = 0.$
8. $36x^5 - 60x^4 - 167x^3 + 52x^2 + 57x - 18 = 0.$

Incommensurable Roots.

494. Location of the Roots. In order to calculate the value of an incommensurable root we must first find a rough approximation to the value of the root; for example, two integers between which it lies. This can generally be accomplished by the aid of the following

Theorem on Change of Sign. *Let two real numbers* **a** *and* **b** *be put for* **x** *in* **f(x)**. *If the resulting values of* **f(x)** *have contrary signs, an odd number of roots of the equation* **f(x) = 0** *lie between* **a** *and* **b**.

As x changes from a to b, passing through all intermediate values, $f(x)$ will change from $f(a)$ to $f(b)$, passing through all intermediate values. Now, in changing from $f(a)$ to $f(b)$, $f(x)$ changes sign.

Hence, $f(x)$ must pass through the value zero. That is, there is some value of x between a and b which causes $f(x)$ to vanish; that is, some root of the equation $f(x)=0$ lies between a and b.

But $f(x)$ may pass through zero more than once. To change sign, $f(x)$ must pass through zero an *odd* number of times; and an odd number of roots must lie between a and b.

Applied to the graph of the equation, since to a root corresponds a point in which the graph meets the axis of x (§ 473), the above simply means that to pass from a point below the axis of x to a point above that axis, we must cross the axis an odd number of times.

In some examples Descartes' Rule of Signs may be of use.

Example. The equation

$$x^4 - 2x^3 - 11x^2 + 6x + 2 = 0$$

has, by Descartes' rule (§ 488), not more than two positive roots and not more than two negative roots.

We find (§ 463),

$$\begin{aligned} f(0) &= +\ \ 2; & f(5) &= +132; \\ f(1) &= -\ \ 4; & f(-1) &= -\ \ 12; \\ f(2) &= -30; & f(-2) &= -\ \ 22; \\ f(3) &= -52; & f(-3) &= +\ \ 20; \\ f(4) &= -22; & f(-4) &= +186. \end{aligned}$$

Hence there are two positive roots, one between 0 and 1, and one between 4 and 5; and two negative roots, one between 0 and -1, and one between -2 and -3.

Let us find more closely a value for the root between 0 and 1. We find $f(0.5) = +2.06+$. Since $f(1) = -4$, the root lies between 0.5 and 1.

Try 0.8: we find $f(0.8) = -0.9+$. Hence the root lies between 0.5 and 0.8.

We find $f(0.7) = +0.4+$. Hence the root lies between 0.7 and 0.8.

In a similar manner we find the root between 0 and -1 to lie between -0.2 and -0.3.

The first significant figures of the roots are accordingly 0.7, 4, -0.2, -2.

Exercise 141.

Determine the first significant figure of each real root of the following equations:

1. $x^3 - x^2 - 2x + 1 = 0.$

2. $x^3 - 5x - 3 = 0.$

3. $x^3 - 5x^2 + 7 = 0.$

4. $x^3 + 2x^2 - 30x + 39 = 0.$

5. $x^3 - 6x^2 + 3x + 5 = 0.$

6. $x^3 + 9x^2 + 24x + 17 = 0.$

7. $x^3 - 15x^2 + 63x - 50 = 0.$

8. $x^4 - 8x^3 + 14x^2 + 4x - 8 = 0.$

495. Horner's Method, Positive Roots. Suppose the first figure of the root to have been found. Any number of remaining figures may be calculated by the method of approximation known as Horner's Method.

We proceed to illustrate the process by an example.

Take the equation

$$x^3 - 6x^2 + 3x + 5 = 0. \tag{1}$$

By § 494 one root of this equation lies between 1 and 2. We proceed to calculate that root.

Diminish the roots by 1 (§ 486):

$$
\begin{array}{cccc|l}
1 & -6 & +3 & +5 & \underline{1} \\
 & \underline{+1} & \underline{-5} & \underline{-2} & \\
 & -5 & -2 & +3 & \\
 & \underline{+1} & \underline{-4} & & \\
 & -4 & -6 & & \\
 & \underline{+1} & & & \\
 & -3 & & &
\end{array}
$$

The transformed equation is, therefore,

$$y^3 - 3y^2 - 6y + 3 = 0. \qquad (2)$$

The roots of equation (2) are each less by 1 than the roots of equation (1). Equation (1) has a root between 1 and 2; equation (2) has, therefore, a root between 0 and 1. Since this root is less than 1, y^3 and y^2 are both less than y. Neglecting these terms, we have

$$-6y + 3 = 0, \text{ or } y = 0.5.$$

At this stage of the process the figure thus obtained will not in general be the correct one. If, however, we neglect only the y^3 term, we obtain

$$-3y^2 - 6y + 3 = 0,$$

or

$$y^2 + 2y - 1 = 0,$$

of which one root is $\sqrt{2} - 1 = 0.4+$.

We can also find the second figure of the root as follows:

Take the first value 0.5.

With this assumed value of y, computing the value of $y^3 - 3y^2$, and substituting, we obtain $6y = 2.375$; whence $y = 0.4$, approximately.

We now diminish the roots of (2) by 0.4:

$$\begin{array}{rrrr|l}
1 & -3 & -6 & +3 & 0.4 \\
 & +0.4 & -1.04 & -2.816 & \\ \hline
 & -2.6 & -7.04 & +\mathbf{0.184} & \\
 & +0.4 & -0.88 & & \\ \hline
 & -2.2 & -\mathbf{7.92} & & \\
 & +0.4 & & & \\ \hline
 & -\mathbf{1.8} & & &
\end{array}$$

The second transformed equation is

$$z^3 - 1.8\,z^2 - 7.92\,z + 0.184 = 0. \qquad (3)$$

The roots of (3) are less by 0.4 than those of (2), and less by 1.4 than those of (1). Equation (2) has a root between 0.4 and 0.5; equation (3) has, therefore, a root between 0 and 0.1.

Since this root is much less than 1, we shall probably obtain a correct value for the next figure of the root by neglecting the z^3 and z^2 terms in equation (3).

This gives $-7.92z + 0.184 = 0$; whence $z = 0.02+$.

Diminish the roots of (3) by 0.02:

$$\begin{array}{rrrr|l}
1 & -1.8 & -7.92 & +0.184 & 0.02 \\
 & +0.02 & -0.0356 & -0.159112 & \\ \hline
 & -1.78 & -7.9556 & +\mathbf{0.024888} & \\
 & +0.02 & -0.0352 & & \\ \hline
 & -1.76 & -\mathbf{7.9908} & & \\
 & +0.02 & & & \\ \hline
 & -\mathbf{1.74} & & &
\end{array}$$

The third transformed equation is

$$u^3 - 1.74\,u^2 - 7.9908\,u + 0.024888 = 0. \qquad (4)$$

The roots of (4) are less by 0.02 than those of (3), and less by 1.42 than those of (1).

Neglecting the u^3 and u^2 terms, we obtain $u = 0.0031+$,

so that to four places of decimals the root of (1) is 1.4231. The process may evidently be continued until the root is calculated to any desired degree of accuracy.

496. We shall now make some observations on the preceding work.

First: If we diminish the roots by a number *less* than the required root, as we do not pass through the root, the sign of the last term remains unchanged throughout the work. The last coefficient but one will always have a sign opposite to that of the last term.

If, in (3), the signs of the last two terms were alike, the value of z would be $-0.02+$. This would show that the value assumed for z was too great, and we should diminish the value of z and make the last transformation again. In *beginning* an example, one is very likely to assume too large a value for the next figure of the root; in solving (2), for instance, the first solution gave $y = 0.5$, and had that value been tried, it would have proved to be too great.

The *first* transformation may, however, change the sign of the last term. Thus, if there had been a root between 0 and 1 in equation (1), diminishing the roots by 1 would have changed the sign of the last term.

Second: In finding the second figure of the root we make use of the last three terms of the first transformed equation instead of the last two terms. Or, we may use the alternative method. One of these methods will generally give the correct figure. In any case we can find the correct figure by another trial.

Any figure after the second is generally found correctly from the last two terms; for, in this case, the root is small and its square and cube so much smaller than the root itself that the terms in which they appear have but slight influence upon the result.

497. It is not necessary to write out the successive transformed equations. When the coefficients of any transformed

equation have been computed, the next figure of the root may be found by dividing the last coefficient by the preceding coefficient, and changing the sign of the quotient.

Thus, in equation (4), the next figure of the root is obtained by dividing 0.024888 by 7.9908.

On this account the last coefficient but one of each transformed equation is called a **trial divisor.**

Sometimes the last coefficient but one in one of the transformed equations is zero. To find the next figure of the root in this case follow the method given for finding the second figure of the root.

The work may now be collected and arranged as follows:

1	-6	$+3$	$+5$	$1.423+$
	$+1$	-5	-2	
	-5	-2	$\mathbf{+3}$	
	$+1$	-4	-2.816	
	-4	$\mathbf{-6}$	$\mathbf{+0.184}$	
	$+1$	-1.04	-0.159112	
	$\mathbf{-3}$	-7.04	$\mathbf{+0.024888}$	
	$+0.4$	-0.88		
	-2.6	$\mathbf{-7.92}$		
	$+0.4$	-0.0356		
	-2.2	-7.9556		
	$+0.4$	-0.0352		
	$\mathbf{-1.8}$	$\mathbf{-7.9908}$		
	$+0.02$			
	-1.78			
	$+0.02$			
	-1.76			
	$+0.02$			
	$\mathbf{-1.74}$			

The broken lines mark the conclusion of each transformation. The numbers in heavy type are the coefficients of the successive transformed equations, the first coefficient of each equation being the same as the first coefficient of the given equation. In this example the first coefficient is 1.

When we have obtained the root to three places of decimals we can generally obtain two or three more figures of the root by simple division.

498. In practice it is convenient to avoid the use of the decimal points. We can do this as follows: multiply the roots of the first transformed equation by 10, the roots of the second transformed equation by 100, and so on. In the last example the first transformed equation will now be

$$y^3 - 30y^2 - 600y + 3000 = 0,$$

and this equation will have a root between 4 and 5. The second transformed equation will now be

$$z^3 - 180z^2 - 79{,}200z + 184{,}000 = 0,$$

and this equation will have a root between 2 and 3. And so on.

Comparing these equations with the equations in § 495, we see that we can avoid the use of the decimal point by adopting the following rule:

When the coefficients of a transformed equation have been obtained, add one cipher to the second coefficient, two ciphers to the third coefficient, and so on. The coefficients and the next figure of the root are now integers. The work proceeds as in § 497.

If the root of the given equation lay between 0 and 1, we should *begin* by multiplying the roots of the given equation by 10.

The complete work of the last example, for six figures of the root, will now be as follows:

1	− 6	+ 3	+ 5	1.42311 +
	+ 1	− 5	− 2	
	− 5	− 2	+ **3000**	
	+ 1	− 4	− 2816	
	− 4	− **600**	+ **184000**	
	+ 1	− 104	− 159112	
	− **30**	− 704	+ **24888000**	
	+ 4	− 88	− 23988033	
	− 26	− **79200**	+ **899967000**	
	+ 4	− 356	− 800138609	
	− 22	− 79556	+ **99828391**	
	+ 4	− 352		
	− **180**	− **7990800**		
	+ 2	− 5211		
	− 178	− 7996011		
	+ 2	− 5202		
	− 176	− **800121300**		
	+ 2	− 17309		
	− **1740**	− 800138609		
	+ 3	− 17308		
	− 1737	− **800155917**		
	+ 3			
	− 1734			
	+ 3			
	− **17310**			
	+ 1			
	− 17309			
	+ 1			
	− 17308			
	+ 1			
	− **17307**			

We can find five more figures of the root by simple division. If we divide 99,828,391 by 800,155,917, we obtain 0.124761, so that the required root to ten places of decimals is 1.4231124761.

The reason is seen by examining the last transformed equation. Write this

$$8.00155917\,w = 0.000099828391 - 1.7307\,w^2 + w^3.$$

As w is about 0.00001, w^2 is about 0.0000000001, and w^3 is still smaller. Hence the error in neglecting the w^2 and w^3 terms is in $8\,w$ about 0.00000000017 and in w about 0.00000000002. The result obtained by division will therefore be true to ten places of decimals.

499. Negative Roots. To avoid the inconvenience of working with negative numbers, when we wish to calculate a negative root, we change the signs of the roots (§ 482), and calculate the corresponding positive roots of the transformed equation.

Thus one root of the equation

$$x^3 - 6\,x^2 + 3\,x + 5 = 0$$

lies between 0 and -1 (§ 494). By Horner's Method we find the corresponding root of

$$x^3 + 6\,x^2 + 3\,x - 5 = 0$$

to be 0.6696+. Hence, the required root of the given equation is $-0.6696+$.

Exercise 142.

Compute for each of the following equations the root of which the first figure is the number in parenthesis opposite the equation. Carry out the work to three places of decimals:

1. $x^3 + 3x - 5 = 0$ $\quad(+1)$.
2. $x^3 - 6x - 12 = 0$ $\quad(+3)$.
3. $x^3 + x^2 + x - 100 = 0.$ $\quad(+4)$.
4. $x^3 + 10x^2 + 6x - 120 = 0$ $\quad(+2)$.
5. $x^3 + 9x^2 + 24x + 17 = 0$ $\quad(-4)$.
6. $x^4 - 12x^3 + 12x - 3 = 0$ $\quad(-1)$.
7. $x^4 - 8x^3 + 14x^2 + 4x - 8 = 0$ $\quad(-0)$.

500. Contraction of Horner's Method. In § 498 the student will see that if we seek only the first six figures of the root, the last six figures of the fourth coefficient of the last transformed equation may be rejected without affecting the result. Those figures of the second and third coefficients which enter into the fourth coefficient only in the rejected figures may also be rejected. Moreover, we may reject all the figures which stand in vertical lines over the figures already rejected.

The work may now be conducted as follows:

1	− 6	+ 3	+ 5	1.42311+
	+ 1	− 5	− 2	
	− 5	− 2	**+ 3000**	
	+ 1	− 4	− 2816	
	− 4	**− 600**	**+ 184000**	
	+ 1	− 104	− 159112	
	− 30	− 704	**+ 24888**	
	+ 4	− 88	− 23991	
	− 26	**− 79200**	+ 897	
	+ 4	− 356	− 800	
	− 22	− 79556	+ 97	
	+ 4	− 352	− 80	
	− 180	**− 79908**		
	+ 2	− 7991		
	− 178	− 6		
	+ 2	− 7997		
	− 176	− 6		
	+ 2	− 8003		
	− 174	− 800		
	− 2	− 80		

The double lines in the first column indicate that beyond this stage of the work the first column disappears altogether.

In the present example we find three figures of the root before we begin to contract. We then contract the work as follows:

Instead of adding ciphers to the coefficients of the transformed equation, we leave the last term as it is; from the last coefficient but one we strike off the last figure; from the last coefficient but two we strike off the last two figures; and so on. In each case we take for the remainder the nearest integer. Thus, in the first column of the preceding example we strike off from 174 the last two figures, and take for the remainder 2 instead of 1.

The contracted process soon reduces to simple division. Thus, in the last example, the last two figures of the root were found by simply dividing 897 by 800.

To insure accuracy in the last figure, the last divisor must consist of at least two figures. Consider the trial divisor at any stage of the work. If we begin to contract, we strike off one figure from the trial divisor *before* finding the next figure of the root. Since the last divisor is to consist of two figures, the contracted process will give us two less figures than there are figures in the trial divisor.

Thus, in § 498, if we begin to contract at the third trial divisor, $-79,908$, we can obtain three more figures of the root; if we begin to contract at the fourth trial divisor, $-8,001,213$, we can obtain five more figures of the root; and so on.

The student should carefully compare the contracted process on page 486 with the uncontracted on page 484.

501. When the root sought is a large number, we cannot find the successive figures of its *integral* portion by dividing the absolute term by the preceding coefficient, because

the neglect of the higher powers, which are in this case large numbers, leads to serious error.

Let it be required to find one root of

$$x^4 - 3x^2 + 11x - 4{,}842{,}624{,}131 = 0. \tag{1}$$

By trial, we find that a root lies between 200 and 300. Diminishing the roots of (1) by 200, we have

$$y^4 + 800y^3 + 239{,}997y^2 + 31{,}998{,}811y - 3{,}242{,}741{,}931 = 0. \tag{2}$$

$$\text{If } y = 60, \quad f(y) = -273{,}064{,}071.$$

$$\text{If } y = 70, \quad f(y) = +471{,}570{,}139.$$

The signs of $f(y)$ show that a root lies between 60 and 70. Diminishing the roots of (2) by 60, we obtain

$$z^4 + 1040z^3 + 405{,}597z^2 + 70{,}302{,}451z - 273{,}064{,}071 = 0. \tag{3}$$

The root of this equation is found by trial to lie between 3 and 4. Diminishing the roots by 3, we may find the remaining figures of the root by the usual process.

502. Any root of a number can be extracted by Horner's Method.

Find the fourth root of 473.

Here $\qquad x^4 = 473,$

or $\qquad x^4 + 0x^3 + 0x^2 + 0x - 473 = 0.$

Calculating the root, $\quad x = 4.66353+.$

If the number be a perfect power, the root will be obtained exactly.

503. From the preceding sections we obtain the following general directions for solving a numerical equation:

I. Find and remove commensurable roots by §§ 492–493, if there are any such roots in the equation.

II. Determine the situation, and then the first figure, of each of the incommensurable roots as in § 494.

III. Calculate the incommensurable roots by Horner's Method.

Exercise 143.

Calculate to six places of decimals the positive roots of the following equations:

1. $x^3 - 3x - 1 = 0.$
2. $x^3 + 2x^2 - 4x - 43 = 0.$
3. $3x^3 + 3x^2 + 8x - 32 = 0.$
4. $2x^3 - 26x^2 + 131x - 202 = 0.$
5. $x^4 - 12x + 7 = 0.$
6. $x^4 - 5x^3 + 2x^2 - 13x + 55 = 0.$

Calculate to six places of decimals the real roots of the following equations, when incommensurable:

7. $x^3 = 35{,}499.$
8. $x^3 = 242{,}970{,}624.$
9. $x^4 = 707{,}281.$
10. $x^5 = 147{,}008{,}443.$
11. $x^3 + 2x + 20 = 0.$
12. $x^3 - 10x^2 + 8x + 120 = 0.$

Sturm's Theorem.

504. The problem of determining the number and situation of the real roots of an equation is completely solved by Sturm's Theorem. In theory Sturm's method is perfect; in practice its application is long and tedious. For this reason, the situation of the roots is in general more easily determined by the methods already given.

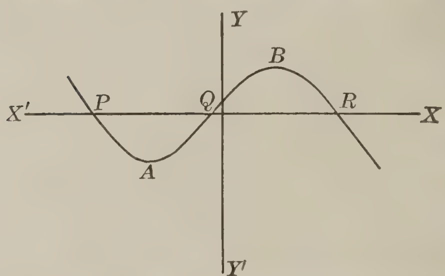
Before passing on to Sturm's Theorem itself we shall prove two preliminary theorems.

505. Sign of f′(x). The function $f'(x)$ is by definition (§ 476) the limit of the ratio of the increment of $f(x)$ to the corresponding increment of x, as the latter approaches 0 as a limit.

If we suppose the increment of x to be always positive, then the corresponding increment of $f(x)$ may be positive or negative. If the increment of $f(x)$ is positive, however small the increment of x may be, then in the limit $f'(x)$ will be positive. But if for very small increments of x, the increment of $f(x)$ is always negative, then $f'(x)$ will be negative.

In other words, $\mathbf{f'(x)}$ *is positive or negative for any value of* $\mathbf{x}$ *according as the function* $\mathbf{f(x)}$ *is increasing or decreasing as* $\mathbf{x}$ *increases from this particular value of* $\mathbf{x}$.

Referring to the graph of $f(x)$, $f'(x)$ will be positive as long as the curve is rising toward the right, and negative when the curve is sinking toward the right. At the highest and lowest points, B and A, the derivative changes its sign, that is, it passes through the value 0 at these points. The values of x corresponding to A and B are roots of the equation $f(x)=0$.



506. Signs of f(x) and f'(x). *Let* α *be any real root of an equation* $\mathbf{f(x)}=0$, *which has no equal roots.*

Let $\mathbf{x}$ *change continuously from* $\alpha - \mathbf{h}$, *a value a little less than* α, *to* $\alpha + \mathbf{h}$, *a value a little greater than* α. *Then* $\mathbf{f(x)}$ *and* $\mathbf{f'(x)}$ *will have unlike signs immediately before* $\mathbf{x}$ *passes through the root, and like signs immediately after* $\mathbf{x}$ *passes through the root.*

For, in the graph, if $f(x)$ is positive just before x passes through a root as at P and R, the curve is sinking to the right, and therefore $f(x)$ is negative, both before and after

the root is passed. But $f(x)$ is positive before we reach the root and negative after we pass it.

Again if $f(x)$ is negative just before we reach a root, as at Q, the curve is rising to the right, and therefore $f'(x)$ is positive on both sides of the root, while $f(x)$ changes its sign from $-$ to $+$ as we pass through Q.

Hence, in both cases $f(x)$ and $f'(x)$ have unlike signs just before we reach a root, and like signs as soon as we have passed a root.

507. Sturm's Functions. The process of finding the H. C. F. of $f(x)$ and $f'(x)$ has been employed (§ 480) in obtaining the multiple roots of the equation $f(x)=0$. We use the same process in Sturm's Method.

Let $f(x)=0$ be an equation which has no multiple roots; let the operation of finding the H. C. F. of $f(x)$ and $f'(x)$ be carried on until the remainder does not involve x, *the sign of each remainder obtained being changed before it is used as a divisor.*

If there is a H. C. F., the equation has multiple roots. Remove them and proceed with the reduced equation.

Represent by $f_2(x), f_3(x), \ldots\ldots f_n(x)$ the several remainders with their signs changed. These expressions with $f'(x)$ are called **Sturm's Functions.**

Now, if D represents the dividend, d the divisor, q the quotient, and R the remainder,

$$D = qd + R.$$

Consequently,
$$\begin{aligned} f(x) &= q_1 f'(x) - f_2(x), \\ f'(x) &= q_2 f_2(x) - f_3(x), \\ f_2(x) &= q_3 f_3(x) - f_4(x), \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \\ f_{n-2}(x) &= q_{n-1} f_{n-1}(x) - f_n(x); \end{aligned}$$

where q_1, q_2, ... q_{n-1} represent the several quotients, or the quotients multiplied by positive integers.

From these identities we have the following:

I. Two consecutive functions cannot vanish for the same value of x.

For example, suppose $f_2(x)$ and $f_3(x)$ to vanish for a particular value of x. Give to x this value in all the identities. By the third identity, $f_4(x)$ will vanish; by the fourth, $f_5(x)$ will vanish; finally, $f_n(x)$ will vanish; but this is contrary to the hypothesis that $f(x) = 0$ has no multiple roots.

II. When we give to x a value which causes any one function to vanish, the adjacent functions have opposite signs.

Thus, if $f_3(x) = 0$, from the third identity $f_2(x) = -f_4(x)$.

508. Sturm's Theorem. We are now in a position to enunciate Sturm's Theorem:

If in the series of functions

$$\mathbf{f}(\mathbf{x}),\ \mathbf{f}'(\mathbf{x}),\ \mathbf{f}_2(\mathbf{x}) \ldots\ldots \mathbf{f}_n(\mathbf{x})$$

we give to **x** *any particular value* **a**, *and determine the number of variations of sign; then give to* **x** *any greater value* **b**, *and determine the number of variations of sign; the number of variations lost is the number of real roots of the equation* $\mathbf{f}(\mathbf{x}) = 0$ *between* **a** *and* **b**.

Let x increase continuously from a to b.

First: Take the case in which x passes through a root of any of the functions $f'(x)$, $f_2(x)$, ... $f_{n-1}(x)$, for example, $f_4(x)$. The adjacent functions in this case have opposite signs. $f_4(x)$ itself changes sign, but this has no effect on

the number of variations; for if just before x passes through the root the signs are $+ + -$, just after x passes through the root they will be $+ - -$, and the number of variations is in each case one.

Hence, there is no change in the number of variations of sign when x passes through a root of any of the functions $f'(x)$, $f_2(x)$, ... $f_{n-1}(x)$.

Second: Take the case in which x passes through a root of $f(x) = 0$. Since $f(x)$ and $f'(x)$ have unlike signs just before x passes through the root, and like signs just after (§ 506), there is one variation lost for each root of $f(x) = 0$.

Hence, the number of real roots between a and b is the number of variations of sign lost as x passes from a to b.

To find the total number of real roots, we take x first very large and negative, and then very large and positive. The sign of each function is then the sign of its first term.

The student may not understand how it is that $f(x)$ and $f'(x)$ always have unlike signs just before x passes through a root.

Let α and β be two consecutive roots of $f(x) = 0$; let h be very small. Suppose that $f(x)$ changes at α from $+$ to $-$; then $f'(\alpha)$ is $-$ (§ 505).

When $x = \alpha - h$, $f(x) = +$, $f'(x)$ is $-$;

$x = \alpha$, $f(x) = 0$, $f'(x)$ is $-$.

As x changes from α to β, $f'(x)$ passes through an odd number of roots (§ 494), and consequently changes sign. Hence, when $x = \beta - h$, $f(x)$ is $-$, $f'(x)$ is $+$; and $f'(x)$ and $f(x)$ again have unlike signs.

509. Example. Determine the number and signs of the real roots of the equation

$$x^4 - 4x^3 + 6x^2 - 12x + 1 = 0.$$

Here $f'(x) = 4x^3 - 12x^2 + 12x - 12.$

Let us take for $f'(x)$, however, the simpler expression

$$x^3 - 3x^2 + 3x - 3.$$

We proceed as if to find the H. C. F., changing the sign of each remainder before using it as a divisor.

1 − 3 + 3 − 3	**1 − 4 + 6 − 12 + 1**	1 − 1
3 − 9 + 9 − 9	1 − 3 + 3 − 3	
3 + 1	− 1 + 3 − 9 + 1	
− 10 + 9	− 1 + 3 − 3 + 3	
− 30 + 27	− 6 − 2	
− 30 − 10	**3 + 1**	1 − 10 + 37
37 − 9		
111 − 27		
111 + 37		
− 64		
+ 64		

The coefficients of the several functions are in heavy type. In the ordinary process of finding the highest common factor we can change signs at pleasure. In finding Sturm's functions we cannot do this as the sign is all important. We can, however, take out any positive factor.

We now have
$$\begin{aligned} f(x) &= x^4 - 4x^3 + 6x^2 - 12x + 1, \\ f'(x) &= x^3 - 3x^2 + 3x - 3, \\ f_2(x) &= 3x + 1, \\ f_3(x) &= +64. \end{aligned}$$

When	$f(x)$	$f'(x)$	$f_2(x)$	$f_3(x)$	
$x = -1000$	+	−	−	+	2 variations.
$x = 0$	+	−	+	+	2 variations.
$x = +1000$	+	+	+	+	0 variations.

Hence the equation has two real positive roots; it must therefore have two imaginary roots.

The real roots will be found by § 494 to lie one between 0 and 1, and one between 3 and 4.

Exercise 144.

Determine by Sturm's Theorem the number and situation of the real roots of the following equations:

1. $x^3 - 4x^2 - 11x + 43 = 0.$
2. $x^3 - 6x^2 + 7x - 3 = 0.$
3. $x^4 - 4x^3 + x^2 + 6x + 2 = 0.$
4. $x^4 - 5x^3 + 10x^2 - 6x - 21 = 0.$
5. $x^4 - x^3 - x^2 + 6 = 0.$
6. $x^4 - 2x^3 - 3x^2 + 10x - 4 = 0.$
7. $x^5 + 2x^4 + 3x^3 + 3x^2 - 1 = 0.$
8. $x^5 + x^3 - 2x^2 + 3x - 2 = 0.$

510. The Cube Roots of Unity. The equation $x^3 = 1$, or $x^3 - 1 = 0$ may be written

$$(x-1)(x^2+x+1) = 0.$$

The roots are $1,\ -\frac{1}{2} + \frac{1}{2}\sqrt{-3},\ -\frac{1}{2} - \frac{1}{2}\sqrt{-3}.$

If either of the imaginary roots is represented by ω, the other is found by actual multiplication to be ω^2; also

$$\omega^2 + \omega + 1 = 0.$$

Every number u has three cube roots. If one of its roots be $u^{\frac{1}{3}}$, the others are $\omega u^{\frac{1}{3}}$ and $\omega^2 u^{\frac{1}{3}}$. For the cube of any one of these is evidently u.

511. The General Cubic. We shall write the general equation of the third degree in the form

$$ax^3 + 3bx^2 + 3cx + d = 0. \qquad (1)$$

Before attempting to solve this we shall transform it into an equation in which the second term is wanting.

Put $z = ax + b$; $\therefore\ x = \dfrac{z - b}{a}$. Substituting this expression for x, and reducing, we obtain

$$z^3 + 3(ac - b^2)z + (a^2d - 3abc + 2b^3) = 0,$$

or, putting $H = ac - b^2$, $G = a^2d - 3abc + 2b^3$,

$$z^3 + 3Hz + G = 0, \tag{2}$$

in (2) put $\quad z = u^{\frac{1}{3}} + v^{\frac{1}{3}}$.

Then

$$(u^{\frac{1}{3}} + v^{\frac{1}{3}})^3 + 3H(u^{\frac{1}{3}} + v^{\frac{1}{3}}) + G = 0,$$

which reduces to

$$u + v + 3(u^{\frac{1}{3}}v^{\frac{1}{3}} + H)(u^{\frac{1}{3}} + v^{\frac{1}{3}}) + G = 0. \tag{3}$$

Since we have assumed but one relation between u and v, we can assume one more relation. Let us assume

$$u^{\frac{1}{3}}v^{\frac{1}{3}} = -H. \tag{4}$$

Equation (3) now reduces to $u + v = -G$. (5)

And (4) may be written $\quad uv = -H^3$. (6)

Eliminating v, we obtain the quadratic

$$u^2 + Gu = H^3, \tag{7}$$

called the *reducing quadratic* of the cubic.

Solving this quadratic, we find

$$\left.\begin{aligned} u &= \frac{-G \pm \sqrt{G^2 + 4H^3}}{2} \\ v &= \frac{-H^3}{u} = \frac{-G \mp \sqrt{G^2 + 4H^3}}{2} \end{aligned}\right\}. \tag{8}$$

Since u and v differ only in having opposite signs before the radical, it will make no difference in the equation $ax + b = z = u^{\frac{1}{3}} + v^{\frac{1}{3}}$ whether the radical be taken with the $+$ sign in u or in v.

Again, when any one of the three values $u^{\frac{1}{3}}$ has been selected, the corresponding value of $v^{\frac{1}{3}}$ must be so taken that (4) is satisfied. Accordingly we obtain just three solutions for z, and consequently for x, corresponding to the three values of $u^{\frac{1}{3}}$. The three values of z are

$$u^{\frac{1}{3}} - \frac{H}{u^{\frac{1}{3}}}, \qquad \omega u^{\frac{1}{3}} - \frac{H}{\omega u^{\frac{1}{3}}}, \qquad \omega^2 u^{\frac{1}{3}} - \frac{H}{\omega^2 u^{\frac{1}{3}}},$$

where $u^{\frac{1}{3}}$ is any one of the three cube roots of u.

The above solution is known as *Cardan's*.

Solve, by Cardan's method,

$$3x^3 + 12x^2 + 12x - 2 = 0.$$

Here $a = 3$, $b = 4$. Putting $z = 3x + 4$, we obtain

$$z^3 - 12z - 34 = 0,$$

$$\therefore\ H = -4,\ G = -34,$$

and the reducing quadratic is

$$u^2 - 34u = -64.$$

Solving, $$u = 2 \text{ or } 32;$$

$$\therefore\ v = -\frac{H^3}{u} = 32 \text{ or } 2.$$

Hence the values of z are

$$\sqrt[3]{2} + \sqrt[3]{32},\ \ \omega\sqrt[3]{2} + \omega^2\sqrt[3]{32},\ \ \omega^2\sqrt[3]{2} + \omega\sqrt[3]{32};$$

or $$\sqrt[3]{2} + 2\sqrt[3]{4},\ \ \omega\sqrt[3]{2} + 2\omega^2\sqrt[3]{4},\ \ \omega^2\sqrt[3]{2} + 2\omega\sqrt[3]{4};$$

and the values of x are

$$\tfrac{1}{3}(\sqrt[3]{2} + 2\sqrt[3]{4} - 4),\ \ \tfrac{1}{3}(\omega\sqrt[3]{2} + 2\omega^2\sqrt[3]{4} - 4),\ \ \tfrac{1}{3}(\omega^2\sqrt[3]{2} + 2\omega\sqrt[3]{4} - 4).$$

512. Discussion of the Solution. Cardan's method furnishes a complete *algebraic* solution of the equation of the third degree, but is of little value in solving *numerical* equations.

It may be shown that the expression $G^2 + 4H^3$ is negative, zero, or positive, according as the three roots of the given equation are real and unequal, two of them equal, or two imaginary and one real. The expression $G^2 + 4H^3$, whose value determines the nature of the roots, is called the **Discriminant** of the cubic. Consider the three cases.

I. *All three roots real and unequal.* In this case, $G^2 + 4H^3$ is negative, and its square root is imaginary. If we put $K^2 = -(G^2 + 4H^3)$, we have

$$ax + b = \left(\frac{-G + K\sqrt{-1}}{2}\right)^{\frac{1}{3}} + \left(\frac{-G - K\sqrt{-1}}{2}\right)^{\frac{1}{3}}.$$

Since there is no general algebraic rule for extracting the cube root of an imaginary expression, the case of three real and unequal roots is known as the *irreducible* case.

II. *Two of the roots equal.* Then $G^2+4H^3=0$, and

$$ax + b = \left(\frac{-G}{2}\right)^{\frac{1}{3}} + \left(\frac{-G}{2}\right)^{\frac{1}{3}}.$$

III. *Two roots imaginary.* Then $G^2 + 4H^3$ is positive, its square root is real, and we have

$$ax+b=\left(\frac{-G+\sqrt{G^2+4H^3}}{2}\right)^{\frac{1}{3}}+\left(\frac{-G-\sqrt{G^2+4H^3}}{2}\right)^{\frac{1}{3}}.$$

Hence, the general solution gives us the roots of a numerical cubic in a form in which their values can be readily computed only in the second and third cases.

In the first case the roots may be calculated by Trigonometry. (See Wentworth's College Algebra, § 535.)

The real roots, however, are more easily found by Horner's method.

CHAPTER XXXVII.

DETERMINANTS.

513. Origin. Solving the two simultaneous equations

$$a_1x + b_1y = c_1,$$

$$a_2x + b_2y = c_2,$$

we obtain

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, \qquad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

Similarly, from the three simultaneous equations

$$a_1x + b_1y + c_1z = d_1,$$

$$a_2x + b_2y + c_2z = d_2,$$

$$a_3x + b_3y + c_3z = d_3,$$

we obtain

$$x = \frac{d_1b_2c_3 - d_1b_3c_2 + d_2b_3c_1 - d_2b_1c_3 + d_3b_1c_2 - d_3b_2c_1}{a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1},$$

with similar expressions for y and z.

The numerators and denominators of these fractions are examples of expressions which often occur in algebraic work, and for which it is therefore convenient to have a special name; such expressions are called **determinants.**

514. Definitions. Determinants are usually written in a compact form, called the *square form.*

Thus, $a_1b_2 - a_2b_1$ is written $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$,

and $$a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1$$

is written $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

This square form is sometimes written in a still more abbreviated form. Thus, the last two determinants are written $|a_1\ b_2|$ and $|a_1\ b_2\ c_3|$. This last notation should, however, always suggest the square form; in any problem it will generally be advisable to write this abbreviated form in the complete square form.

The individual symbols a_1, a_2, b_1, b_2, ..., are called **elements.**

A horizontal line of elements is called a **row**; a vertical line a **column.**

The two lines a_1, b_2, c_3 and a_3, b_2, c_1 are called **diagonals**; the first the **principal diagonal,** the second the **secondary diagonal.**

The **order** of a determinant is the number of elements in a row or column.

Thus, the last two determinants are of the second and third orders, respectively.

The expression of which the square form is an abbreviation is called the **expanded form,** or simply the **expansion,** of the determinant.

The several terms of the expansion are called **terms** of the determinant.

Thus the expansion of $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ is $a_1b_2 - a_2b_1$.

REMARK. By some writers *constituent* is used where we use *element*, and *element* where we use *term*.

515. General Definition. In general, a determinant of the nth order is an expression involving n^2 elements arranged in n rows of n elements each; the expansion, that is, the expression for which the square form is an abbreviation, being found as follows:

Form all the possible products of n elements each that can be formed by taking one, *and only one*, element from each row, and one, *and only one*, element from each column; prefix to each of the products thus formed either $+$ or $-$ (which sign is to be determined by a rule to be given in the following sections), and take the sum of all these products.

NOTE. Nearly all the properties of determinants can be obtained directly from this definition and the rule of signs (§ 518 or § 519). This will be the method followed in the present chapter. It is therefore of the utmost importance that the student should thoroughly understand the present and the four following sections.

516. Inversions of Order. In any particular determinant the letters and subscripts in the principal diagonal are said to be in the *natural order.* If the letters, or subscripts, are taken in any other order, there will be one or more *inversions* of order.

Thus, if 1, 2, 3, 4, 5 be the natural order, in the order 2, 3, 5, 1, 4, there will be four inversions: 2 before 1, 3 before 1, 5 before 1, 5 before 4.

Similarly, if a, b, c, d be the natural order, in the order b, d, a, c, there will be three inversions: b before a, d before a, d before c.

517. In any series of integers (or letters) let two adjacent integers (or letters) be interchanged; then, the number of inversions is either increased or diminished by one.

For example, in the series 6 2 [5 1] 4 3 7, interchange 5 and 1.

We now have 6 2 [1 5] 4 3 7.

The inversions of 5 and 1 with the integers before the group are the same in both series.

The inversions of 5 and 1 with the integers after the group are the same in both series.

In the first series 5 1 is an inversion; in the second series 1 5 is not.

Hence, the interchanging of 5 and 1 diminishes the number of inversions by one.

Similarly, for any case.

518. Signs of the Terms. The principal diagonal term always has a + sign.

To find the sign of any other term: Add together the number of inversions among the letters, and the number of inversions among the subscripts. If the total number is *even*, the sign of the term is +; if *odd*, −.

Thus, in the determinant $|a_1\ b_2\ c_3\ d_4|$ consider the term $c_2a_3d_4b_1$. There are in $c\ a\ d\ b$ three inversions; in 2 3 4 1 three inversions; the total is six, an even number, and the sign of the term is +.

519. In practice the sign of a term is easily found by one of the following special rules:

RULE I. *Write the elements of the term in the natural order of letters; if the number of inversions among the subscripts is even, the sign of the term is +; if odd, −.*

RULE II. *Write the elements in the natural order of subscripts; if the number of inversions among the letters is even, the sign of the term is +; if odd, −.*

Thus, in the determinant $|a_1\ b_2\ c_3\ d_4|$ consider the term $c_2a_3d_4b_1$. Writing the elements in the order of letters, we have $a_3b_1c_2d_4$. There are two inversions, viz.: 3 before 1, and 3 before 2; and the sign of the term is +. Or, write the elements in the order of subscripts, $b_1c_2a_3d_4$. There are two inversions, viz.: b before a, and c before a; and the sign of the term is +.

That these special rules give the same sign as the general rule of § 518 may be seen as follows:

Consider the term $c_2a_3d_4b_1$. Its sign is determined by the total number of inversions in the two series $\begin{matrix} c\ a\ d\ b \\ 2\ 3\ 4\ 1 \end{matrix}$. Bring a_3 to the first position; this interchanges in the two series c and a, 2 and 3. In each series the number of inversions is increased or diminished by one (§ 517), and the total is therefore increased or diminished by an even number.

Interchange b_1 and d_4, then interchange b_1 and c_2; this brings b_1 to the second place, and the letters into the natural order. As before, the total number of inversions is changed by an even number.

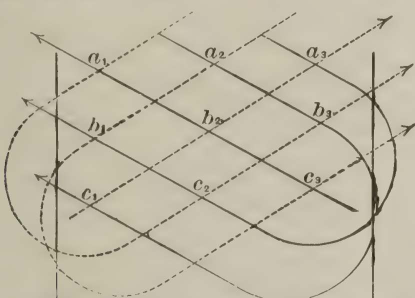
The term is now written $a_3b_1c_2d_4$, and the number of inversions differs by an even number from that found by the general rule of § 518. Hence, the sign given by Rule I. agrees with the sign given by the general rule.

520. If all the elements in any row (or column) are zero, the determinant is zero. For every term contains one of the zeros from this row (or column) (§ 515), and therefore every term of the determinant is zero.

A determinant is unchanged if the rows are changed to columns and the columns to rows. For the rules (§§ 515, 518) are unchanged if "row" is changed to "column" and "column" to "row."

Thus, $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

521. A determinant of the *third order* may be conveniently expanded as follows:



Three elements connected by a full line form a positive term; three elements connected by a dotted line form a negative term. The expansion obtained from the diagram is

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1,$$

which agrees with § 518.

There is no simple rule for expanding determinants of orders higher than the third.

Exercise 145.

Prove the following relations by expanding:

1. $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = -\begin{vmatrix} a_2 & a_1 \\ b_2 & b_1 \end{vmatrix} = \begin{vmatrix} b_2 & b_1 \\ a_2 & a_1 \end{vmatrix}.$

2. $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \begin{vmatrix} a_3 & a_2 & a_1 \\ c_3 & c_2 & c_1 \\ b_3 & b_2 & b_1 \end{vmatrix} = -\begin{vmatrix} b_1 & c_1 & a_1 \\ b_3 & c_3 & a_3 \\ b_2 & c_2 & a_2 \end{vmatrix}.$

Find the values of:

3. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 4 \\ 3 & 4 & 5 \end{vmatrix}.$ **4.** $\begin{vmatrix} 3 & 2 & 4 \\ 7 & 6 & 1 \\ 5 & 3 & 8 \end{vmatrix}.$ **5.** $\begin{vmatrix} 4 & 5 & 2 \\ -1 & 2 & -3 \\ 6 & -4 & 5 \end{vmatrix}.$

6. Count the inversions in the series:

5 4 1 3 2. 7 5 1 4 3 6 2. $d\ a\ c\ e\ b$.

4 1 5 2 3. 6 5 4 2 1 3 7. $c\ e\ b\ d\ a$.

7. In the determinant $|\, a_1\, b_2\, c_3\, d_4\, e_5 \,|$ find the signs of the following terms:

$a_1b_4c_5d_3e_2$. $a_5b_1c_3d_4e_2$. $e_1c_4a_2b_5d_3$.

$a_2b_5c_3d_1e_4$. $b_4c_5a_1e_3d_2$ $c_1a_5b_3e_4d_2$.

8. Write, with their proper signs, all the terms of the determinant $|\, a_1\, b_2\, c_3\, d_4 \,|$.

9. Write with their proper signs, all the terms of the determinant $|\, a_1\, b_2\, c_3\, d_4\, e_5 \,|$ which contain both a_1 and b_4; all the terms which contain both b_3 and e_5.

Expand the determinants:

10. $\begin{vmatrix} a & b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & a & a & b \\ 0 & b & b & a \end{vmatrix}$ **11.** $\begin{vmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ a & a & b & b \\ b & b & a & a \end{vmatrix}$ **12.** $\begin{vmatrix} a & b & c & 0 \\ c & a & b & 0 \\ b & c & a & 0 \\ a & b & c & 1 \end{vmatrix}.$

522. Number of Terms. Consider a determinant of the nth order.

In forming a term we can take from the first row any one of n elements; from the second row any one of $n-1$ elements; and so on. From the last row we can take only the one remaining element.

Hence, the full number of terms is $n(n-1) \ldots\ldots 1$, or $\underline{|n}$.

523. Interchange of Columns (or Rows). *If two adjacent columns of a determinant* Δ *are interchanged, the determinant thus obtained is* $-\Delta$.

For example, consider the determinants

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}, \qquad \Delta' = \begin{vmatrix} a_1 & a_3 & a_2 & a_4 \\ b_1 & b_3 & b_2 & b_4 \\ c_1 & c_3 & c_2 & c_4 \\ d_1 & d_3 & d_2 & d_4 \end{vmatrix}.$$

The individual elements in any row or column of Δ' are the same as those of some row or column of Δ, the only difference being in the arrangement of elements. Since every term of each determinant contains one, and only one, element from each row and column, every term of Δ' must, disregarding the sign, be a term of Δ.

Now the sign of any particular term of Δ' is found from a series (§ 519, Rule I.) in which 3 2 is the natural order. The sign of the term of Δ which contains the same elements is found from a series in which 3 2 is regarded as an inversion. Consequently every term which in Δ' has a $+$ sign has in Δ a $-$ sign, and *vice versa* (§ 517).

Therefore $$\Delta' = -\Delta.$$

Similarly if any two adjacent columns or rows of *any* determinant are interchanged.

524. *In any determinant Δ, if a particular column is carried over* m *columns, the determinant obtained is* $(-1)^m\Delta$.

For, successively interchange the column in question with the adjacent column until it occupies the desired position. There will be m interchanges made, and since there will be m changes of sign (§ 523), the new determinant will be $(-1)^m\Delta$.

Similarly for a particular row.

525. *In any determinant Δ if any two columns are interchanged, the determinant thus obtained is* $-\Delta$.

Let there be m columns between the columns in question.

Bring the second column before the first. The second column will be carried over $m+1$ columns, and the determinant obtained is $(-1)^{m+1}\Delta$ (§ 524).

Bring the first column to the original position of the second. The first column will be carried over m columns, and the determinant obtained is $(-1)^m(-1)^{m+1}\Delta$, or $(-1)^{2m+1}\Delta$.

Since $2m+1$ is always an odd number, this is $-\Delta$.

Similarly for two rows.

$$\text{Thus,}\quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} a_3 & a_2 & a_1 \\ b_3 & b_2 & b_1 \\ c_3 & c_2 & c_1 \end{vmatrix} = \begin{vmatrix} a_3 & a_2 & a_1 \\ c_3 & c_2 & c_1 \\ b_3 & b_2 & b_1 \end{vmatrix}.$$

526. Useful Properties. *If two columns of a determinant are identical, the determinant vanishes.*

For, let Δ represent the determinant.

Interchanging the two identical columns ought to change Δ into $-\Delta$. But since the two columns are identical, the determinant is unchanged.

$$\therefore\ \Delta = -\Delta,\quad 2\Delta = 0,\quad \Delta = 0.$$

Similarly, if two rows are identical.

527. *If all the elements in any column be multiplied by any number* **m**, *the determinant will be multiplied by* **m.**

For every term contains one, and only one, element from the column in question. Hence every term, and consequently the whole determinant, is multiplied by m.

Similarly for a row.

Thus, $$\begin{vmatrix} ma_1 & a_2 & a_3 \\ mb_1 & b_2 & b_3 \\ mc_1 & c_2 & c_3 \end{vmatrix} = m \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Again, $$\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \frac{1}{abc} \begin{vmatrix} abc & a^2 & a^3 \\ bca & b^2 & b^3 \\ cab & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}.$$

528. If each of the elements in a column is the sum of two numbers, the determinant may be expressed as the sum of two determinants.

Thus, $$\begin{vmatrix} a_1+\alpha & a_2 & a_3 \\ b_1+\beta & b_2 & b_3 \\ c_1+\gamma & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha & a_2 & a_3 \\ \beta & b_2 & b_3 \\ \gamma & c_2 & c_3 \end{vmatrix}.$$

For, consider any term, as $(a_1+\alpha)b_2c_3$. This may be written $a_1b_2c_3+\alpha b_2c_3$. Hence, every term of the first determinant is the sum of a term of the second determinant and a term of the third determinant. Consequently the first determinant is the sum of the other two determinants.

Similarly for any other case.

529. If the elements in any column (or row) are multiplied by any number m, and added to, or subtracted from, the corresponding elements in any other column (or row), the determinant is unchanged.

Thus, $$\begin{vmatrix} a_1 \pm ma_2 & a_2 & a_3 \\ b_1 \pm mb_2 & b_2 & b_3 \\ c_1 \pm mc_2 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \pm \begin{vmatrix} ma_2 & a_2 & a_3 \\ mb_2 & b_2 & b_3 \\ mc_2 & c_2 & c_3 \end{vmatrix}.$$

The last determinant may be written

$$\pm m \begin{vmatrix} a_2 & a_2 & a_3 \\ b_2 & b_2 & b_3 \\ c_2 & c_2 & c_3 \end{vmatrix}, \text{ and therefore vanishes (§ 526).}$$

Hence, we have only the first determinant on the right-hand side.

Similarly for any other case.

This process may be applied simultaneously to two or more columns (or rows); but in this case care must be taken not to make two columns (or rows) identical (§ 526).

The last property is of great use in reducing determinants to simpler forms.

530. Examples.

(1) $$\begin{vmatrix} b+c & a & 1 \\ c+a & b & 1 \\ a+b & c & 1 \end{vmatrix} = \begin{vmatrix} b+c+a & a & 1 \\ c+a+b & b & 1 \\ a+b+c & c & 1 \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix} = 0.$$

Begin by adding the second column to the first.

(2) $$\begin{vmatrix} 14 & 15 & 11 \\ 21 & 22 & 16 \\ 23 & 29 & 17 \end{vmatrix} = \begin{vmatrix} 3 & 4 & 11 \\ 5 & 6 & 16 \\ 6 & 12 & 17 \end{vmatrix} = 2 \begin{vmatrix} 3 & 2 & 11 \\ 5 & 3 & 16 \\ 6 & 6 & 17 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 3 & 2 & 2 \\ 5 & 3 & 1 \\ 6 & 6 & -1 \end{vmatrix} = 2(19) = 38.$$

Begin by subtracting the third column from the first and second columns. Then take out the factor 2, subtract 3 times the first column from the third, and multiply out the result by § 521.

Exercise 146.

Show that

1. $\begin{vmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{vmatrix} = 2abc.$ 2. $\begin{vmatrix} a & b & a \\ b & a & a \\ b & a & b \end{vmatrix} = -(a-b)^2(a+b).$

3. $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc.$

4. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a).$

Find the values of

5. $\begin{vmatrix} 3 & 5 & 7 \\ 2 & 1 & 3 \\ 4 & 3 & 7 \end{vmatrix}.$ 6. $\begin{vmatrix} 2 & 13 & 20 \\ 3 & 9 & 18 \\ 5 & 10 & 23 \end{vmatrix}.$ 7. $\begin{vmatrix} 19 & 13 & 16 \\ 25 & 16 & 28 \\ 28 & 10 & 19 \end{vmatrix}.$

Show that

8. $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ac \\ c & c^2 & ab \end{vmatrix} = -(a-b)(b-c)(c-a)(ab+bc+ca).$

9. $\begin{vmatrix} a+2b & a+4b & a+6b \\ a+3b & a+5b & a+7b \\ a+4b & a+6b & a+8b \end{vmatrix} = 0.$

10. $\begin{vmatrix} (a+b)^2 & c^2 & c^2 \\ a^2 & (b+c)^2 & a^2 \\ b^2 & b^2 & (c+a)^2 \end{vmatrix} = 2abc(a+b+c)^3.$

531. **Minors.** If one row and one column of a determinant be erased, a new determinant of order one lower than the given determinant is obtained. This determinant is called a **first minor** of the given determinant.

Similarly, by erasing two rows and two columns we obtain a **second minor** and so on.

Thus, in the determinant $|a_1\ b_2\ c_3|$, erasing the second row and third column, we obtain the first minor $\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$. This minor is said to correspond to the element b_3, and is generally represented by Δ_{b_3}; so that, in this case, $\Delta_{b_3} = \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

In general, to every element corresponds a first minor obtained by erasing the row and column in which the given element stands.

532. Theorem. *If all the elements of the first row after the first element are zeros, the determinant reduces to* $a_1\Delta_{a_1}$.

Consider the determinant

$$\Delta = \begin{vmatrix} a_1 & 0 & 0 & 0 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}.$$

Every term of Δ contains one, and only one, element from the first row; and all the terms that do not contain a_1 contain one of the zeros, and therefore vanish. The terms that contain a_1 contain no other element from the first row or column, and, consequently, contain one, and only one, element from each row and column of the determinant

$$\begin{vmatrix} b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \\ d_2 & d_3 & d_4 \end{vmatrix}, \text{ or } \Delta_{a_1}.$$

Hence, disregarding the sign, each term of Δ consists of a_1 multiplied into a term of Δ_{a_1}.

Take any particular term of Δ, as $a_1b_4c_3d_2$; the sign is fixed (§ 519, Rule I.) by the number of inversions in the

series 1 4 3 2; the sign of the term $b_4c_3d_2$ of Δ_{a_1} is fixed by the number of inversions in the series 4 3 2. Adding a_1 makes no new inversions among either the letters or the subscripts. Consequently the sign of the term in Δ is the same as the sign of the term in $a_1\Delta_{a_1}$.

Since this is true of every term of Δ, we have

$$\Delta = a_1\Delta_{a_1}.$$

Similarly for any determinant of like form.

533. Terms containing an Element. From § 532 it appears that the sum of the terms which contain a_1 may be written $a_1\Delta_{a_1}$. For, no one of the terms which contain a_1 can contain any one of the elements a_2, a_3, a_4, ..., and these terms are therefore unchanged if for a_2, a_3, a_4, ... in the given determinant we put zeros.

If we carry the second column over the first, the determinant is changed to $-\Delta$. By § 532 the sum of the terms of $-\Delta$ which contain a_2 is $a_2\Delta_{a_2}$, and the sum of the corresponding terms of Δ is therefore $-a_2\Delta_{a_2}$.

In general, for the element of the pth row and qth column, we shall have to carry the pth row over $p-1$ rows, and the qth column over $q-1$ columns in order to bring the element in question to the first row and first column. The new determinant is Δ if $p+q-2$ is *even*, and is $-\Delta$ if $p+q-2$ is *odd* (§ 524). Consequently, the sum of the terms of Δ which contain the element of the pth row and qth column is the product of that element by its minor; the sign being $+$ if $p+q$ is *even*, and $-$ if $p+q$ is *odd*.

Thus, in $\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$ the sum of the terms which contain c_3 is $c_3\Delta_{c_3}$.

Here $p=3$, $q=3$, and $p+q$ is *even*.

534. Co-factors. Since every term contains one element from each row and column, if we add together the sum of the terms containing a_1, the sum of the terms containing a_2, and so on, we shall obtain the whole expansion of the given determinant.

Thus, in the determinant $|a_1\, b_2\, c_3\, d_4|$,

$$\Delta = a_1\Delta_{a_1} - a_2\Delta_{a_2} + a_3\Delta_{a_3} - a_4\Delta_{a_2}.$$

The expressions Δ_{a_1}, $-\Delta_{a_2}$, Δ_{a_3}, $-\Delta_{a_4}$ are called the **co-factors** of the several elements a_1, a_2, a_3, a_4, and are generally represented by A_1, A_2, A_3, A_4.

Hence, in the case of $|a_1\, b_2\, c_3\, d_4|$, we may write

$$\begin{aligned}\Delta &= a_1A_1 + a_2A_2 + a_3A_3 + a_4A_4,\\ &= b_1B_1 + b_2B_2 + b_3B_3 + b_4B_4,\\ &= a_1A_1 + b_1B_1 + c_1C_1 + d_1D_1,\end{aligned}$$

and so on. Similarly for any determinant.

535. Theorem. *If the elements in any row are multiplied by the co-factors of the corresponding elements in another row, the sum of the products vanishes.*

Thus, in the determinant $|a_1\, b_2\, c_3\, d_4|$,

$$b_1B_1 + b_2B_2 + b_3B_3 + b_4B_4 = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}.$$

No one of the co-factors B_1, B_2, B_3, B_4, contains any of the elements b_1, b_2, b_3, b_4. These co-factors will, consequently, be unaffected if in the above identity we change b_1, b_2, b_3, b_4 to a_1, a_2, a_3, a_4. This gives

$$a_1B_1 + a_2B_2 + a_3B_3 + a_4B_4 = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_1 & a_2 & a_3 & a_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = 0.$$

Similarly for any other case.

536. Evaluation of Determinants. By using § 517, § 519, and § 534 we can readily obtain the value of any numerical determinant.

Ex. Evaluate $\begin{vmatrix} 3 & 1 & 4 & 1 \\ 1 & 3 & 2 & 1 \\ 2 & 1 & 3 & 3 \\ 4 & 3 & 2 & 3 \end{vmatrix}.$

From the first row subtract 3 times the second, from the third twice the second, from the fourth 4 times the second. The result is

$$\begin{vmatrix} 0 & -8 & -2 & -2 \\ 1 & 3 & 2 & 1 \\ 0 & -5 & -1 & 1 \\ 0 & -9 & -6 & -1 \end{vmatrix}$$

which, by § 534, reduces to

$$-\begin{vmatrix} -8 & -2 & -2 \\ -5 & -1 & 1 \\ -9 & -6 & -1 \end{vmatrix} \text{ or } \begin{vmatrix} 8 & 2 & 2 \\ 5 & 1 & -1 \\ 9 & 6 & 1 \end{vmatrix} = 70 \text{ (§ 521).}$$

537. Simultaneous Equations. Consider the simultaneous equations

$$a_1x + b_1y + c_1z = k_1,$$
$$a_2x + b_2y + c_2z = k_2,$$
$$a_3x + b_3y + c_3z = k_3.$$

Write the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, and let A_1, A_2, B_1, B_2, etc., be the co-factors in this determinant.

Multiply the first equation by A_1, the second by A_2, the third by A_3, and add. The result is

$$(a_1A_1 + a_2A_2 + a_3A_3)x = k_1A_1 + k_2A_2 + k_3A_3,$$

since (§ 535), $\qquad b_1A_1 + b_2A_2 + b_3A_3 = 0,$

and $\qquad c_1A_1 + c_2A_2 + c_3A_3 = 0.$

Hence (§ 534), we see that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} x = \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}, \text{ or } x = \frac{|k_1\, b_2\, c_3|}{|a_1\, b_2\, c_3|}.$$

Also, $$y = \frac{|a_1\, k_2\, b_3|}{|a_1\, b_2\, c_3|}; \qquad z = \frac{|a_1\, b_2\, k_3|}{|a_1\, b_2\, c_3|}.$$

Similarly for any set of simultaneous equations of the first degree.

Exercise 147.

1. In the determinant $|a_1\, b_2\, c_3\, d_4$ write the co-factors of a_3, b_2, b_4, c_1, c_4, d_2, d_3.

2. Express as a single determinant

$$\begin{vmatrix} e & f & g \\ f & h & k \\ g & k & l \end{vmatrix} + \begin{vmatrix} b & e & g \\ c & f & k \\ d & g & l \end{vmatrix} + \begin{vmatrix} b & g & f \\ c & k & h \\ d & l & k \end{vmatrix} + \begin{vmatrix} b & f & e \\ c & h & f \\ d & k & g \end{vmatrix}.$$

Expand:

3. $\begin{vmatrix} a & b & b & a \\ b & a & a & b \\ a & a & b & b \\ 0 & a & b & b \end{vmatrix}$. **4.** $\begin{vmatrix} 0 & d & d & d \\ a & 0 & a & a \\ b & b & 0 & b \\ c & c & c & 0 \end{vmatrix}$. **5.** $\begin{vmatrix} 1 & a & a & a \\ 1 & b & a & a \\ 1 & a & b & a \\ 1 & a & a & b \end{vmatrix}$.

Find the value of:

6. $\begin{vmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{vmatrix}$. **7.** $\begin{vmatrix} 3 & 2 & 1 & 4 \\ 15 & 29 & 2 & 14 \\ 16 & 19 & 3 & 17 \\ 33 & 39 & 8 & 38 \end{vmatrix}$. **8.** $\begin{vmatrix} 2 & 1 & 3 & 4 \\ 7 & 4 & 5 & 9 \\ 3 & 3 & 6 & 2 \\ 1 & 7 & 7 & 5 \end{vmatrix}$.

Solve the equations:

9. $\left.\begin{aligned} 3x - 4y + 2z &= 1 \\ 2x + 3y - 3z &= -1 \\ 5x - 5y + 4z &= 7 \end{aligned}\right\}.$ **10.** $\left.\begin{aligned} 4x - 7y + z &= 16 \\ 3x + y - 2z &= 10 \\ 5x - 6y - 3z &= 10 \end{aligned}\right\}.$

CHAPTER XXXVIII.

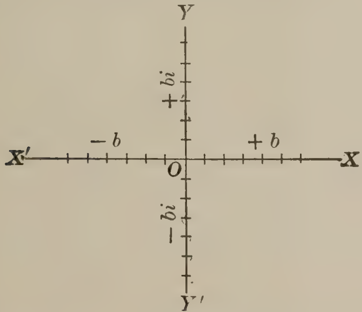
COMPLEX NUMBERS.

538. Representation of Pure Imaginaries. Represent $\sqrt{-1}$ by i. Assuming the commutative and associative laws,

$$\left.\begin{aligned} i \times b, \text{ that is, } (+i)b &= +bi; \\ i \times i \times b = i^2 b = (-1)b &= -b; \\ i \times i \times i \times b = i^3 b = (-i)b &= -bi; \\ i \times i \times i \times i \times b = i^4 b = (+1)b &= +b. \end{aligned}\right\} \qquad \S\ 234.$$

Hence, two multiplications by i change b to $-b$; that is, two multiplications by i turn the line represented by b through 180°; and four multiplications by i turn the line represented by b through 360°; and so on.

We may therefore consistently assume that *one* multiplication by i turns the representative line through 90°, *three* multiplications by i through 270°, and so on.



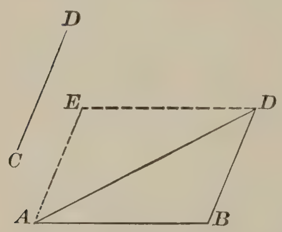
If then we draw through O, the zero point of the line of real numbers, a line YY' perpendicular to XX', all pure

imaginaries will be represented by lengths on this line, just as all real numbers are represented by lengths on XX'.

539. Vectors. A directed straight line of definite length is called a **vector.**

Two parallel vectors which have the same length, and extend in the same direction, are said to be *equal vectors.*

540. Vector Addition. To *add* a vector $\overline{CD}$ to a vector $\overline{AB}$, we place C on B, keeping $\overline{CD}$ parallel to its original position, and draw $\overline{AD}$. Then,



$$\overline{AD} = \overline{AB} + \overline{BD}$$
$$= \overline{AB} + \overline{CD}. \qquad (1)$$

The addition here meant by the sign $+$ is not addition of numbers, but addition of *vectors*, generally called *geometric addition.* It is identical with the composition of forces.

From the dotted lines in the figure, and the known properties of a parallelogram, it is seen that

$$\overline{AD} = \overline{CD} + \overline{AB}. \qquad (2)$$

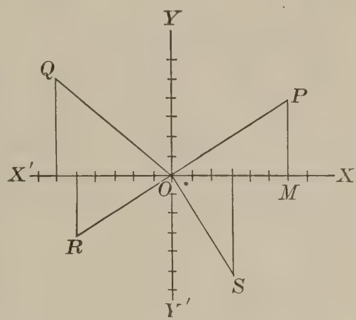
From (1) and (2), $\overline{AB} + \overline{CD} = \overline{CD} + \overline{AB}$.

Consequently, vector addition is *commutative* (§ 31). It is easily seen that it is also *associative* (§ 32).

541. Complex Numbers. A complex number in general consists of a real part and an imaginary part, and may be written (§ 235) in the *typical form* $a + bi$, where a and b are both real.

If we understand the sign $+$ to indicate *geometric* addition, we shall obtain the vector which represents $a + bi$ as follows:

Lay off a on the axis of reals from O to M. From M draw the vector $\overline{MP}$ to represent bi. Then the vector $\overline{OP}$ is the geometric sum of the vectors $\overline{OM}$ and $\overline{MP}$, and represents the complex number $a + bi$.



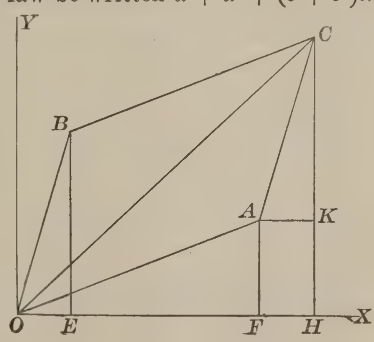
In the figure, the vectors $\overline{OP}$, $\overline{OQ}$, $\overline{OR}$, $\overline{OS}$, respectively, represent the complex numbers $6 + 4i$, $-6+5i$, $-5-3i$, $3-5i$.

In the complex number $a + bi$, a and bi are represented by vectors. Now vector addition is commutative. Consequently, $a + bi = bi + a$.

This is also evident from the figure.

The expression $a + bi$ is the general expression for all numbers. This expression is zero when $a = 0$ and $b = 0$; is a real number when $b = 0$; a pure imaginary when $a = 0$; a complex number when a and b both differ from 0.

542. Addition of Complex Numbers. Let $a + bi$ and $a' + b'i$ be two complex numbers. Their sum, $a + bi + a' + b'i$, may by the commutative law be written $a + a' + (b + b')i$.



Let $\overline{OA}$ and $\overline{OB}$ be the representative vectors of $a + bi$ and $a' + b'i$. Take $\overline{AC} =$ and $\parallel$ to $\overline{OB}$; then, $\overline{OC} = \overline{OA} + \overline{OB}$.

Draw the other lines in the figure.

Then,

$OH = OF + FH$

$\quad = OF + OE = a + a'$,

and $\qquad HC = FA + KC = FA + EB = bi + b'i.$

$$\therefore \overline{OC} = a + a' + (b + b')i = (a + bi) + (a' + b'i).$$

But $\qquad \overline{OC} = \overline{OA} + \overline{OB}.$

Consequently, *the geometric sum of the vectors of two complex numbers is the vector of their sum.*

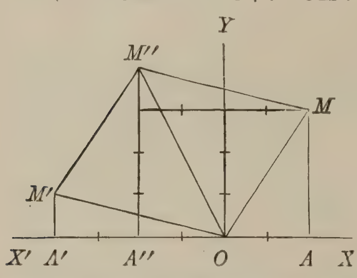
Since vector addition is commutative, it follows that the addition of complex numbers is *commutative.*

The sum of two complex numbers is the geometric sum of the sum of the real parts and the sum of the imaginary parts of the two numbers.

The preceding may be made clearer by a particular example.

Find the sum of $2 + 3i$ and $-4 + i$.

$2 + 3i = OM$ and $-4 + i = OM'$. If now we proceed from M, the extremity of OM, in the direction of OM' as far as the absolute value of OM', we reach the point M''.



Hence, $OM'' = -2 + 4i$, the sum of the two given complex numbers.

The same result is reached if we first find the value of $2 + (-4) = -2$. That is, if we count from O two real units to A'', and add to this sum $3i + i = 4i$; that is, count four imaginary units from A'' on the perpendicular $A''M''$.

543. Modulus and Amplitude. Any complex number, $a + bi$, can be written in the form

$$\sqrt{a^2 + b^2}\left(\frac{a}{\sqrt{a^2 + b^2}} + \frac{b}{\sqrt{a^2 + b^2}}i\right).$$

The expressions $\dfrac{a}{\sqrt{a^2 + b^2}}$ and $\dfrac{b}{\sqrt{a^2 + b^2}}$ may be taken

as the sine and cosine of some angle ϕ, since they satisfy the equation

$$\cos^2\phi + \sin^2\phi = 1.$$

If we put $r = \sqrt{a^2+b^2}$, the complex number may be written

$$r(\cos\phi + i\sin\phi).$$

Since $r = \sqrt{a^2+b^2}$, the sign of r is indeterminate. We shall, however, take r always *positive.*

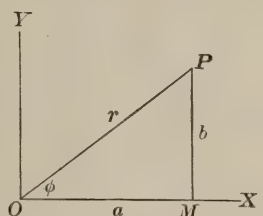
The positive number r is called the **modulus,** the angle ϕ the **amplitude,** of the complex number $a+bi$.

Let OP be the representative vector of $a+bi$. Since r is the positive value of $\sqrt{a^2+b^2}$, it is evident that r is the *length* of OP. Since

$$\cos\phi = \frac{a}{\sqrt{a^2+b^2}} = \frac{a}{r} = \frac{OM}{OP},$$

and

$$\sin\phi = \frac{b}{\sqrt{a^2+b^2}} = \frac{b}{r} = \frac{MP}{OP},$$



it follows that ϕ is the angle MOP.

The above is easily seen to hold true when a and b are one or both negative.

The modulus of a real number is its absolute value; the amplitude is 0 if the number is positive, 180° if the number is negative.

The modulus of a pure imaginary bi is b; the amplitude is 90° if b is positive, 270° if b is negative.

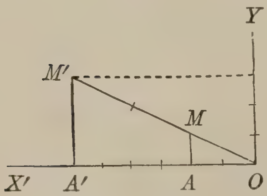
544. Since the sum of the lengths of two sides of a triangle is greater than the length of the third side, it follows from §§ 540, 542, that *the modulus of the sum of two complex numbers is less than the sum of the moduli.*

In one case, however, that in which the representative vectors are collinear, the modulus of the sum is *equal* to the sum of the moduli.

545. Multiplication by Real Numbers. Let $a+bi$ be any complex number. If the representative vector be multiplied by any real number c, it is easily seen from a figure that the product is $ca+cbi$.

Therefore, $$c(a+bi)=ca+cbi.$$

It follows that the multiplication of a complex number by a real number is *distributive*.



Ex. To multiply $-2+i$ by 3: Take $OA=-2$ on OX', and erect at A the perpendicular $AM=i$. Then $OM=-2+i$; and, by taking OM three times, the result is $OM'=-6+3i$, the product of $(-2+i)$ by 3.

546. Multiplication by Pure Imaginaries. We have seen (§ 538) that multiplying a real number, or a pure imaginary, by i turns that number through 90°. Let us consider the effect of multiplying a complex number by i.

By the commutative, associative, and distributive laws,

$$i\times r(\cos\phi+i\sin\phi)=r(i\cos\phi-\sin\phi)$$
$$=r(-\sin\phi+i\cos\phi)$$
by Trigonometry, $$=r[\cos(90^\circ+\phi)+i\sin(90^\circ+\phi)].$$

Here, also, the effect of multiplying by i is to increase ϕ to $\phi+90^\circ$; that is, to turn the representative vector in the positive direction through an angle of 90°.

The effect of multiplying by a pure imaginary bi will be to turn the complex number through a positive angle of 90°, and also to multiply the modulus by b.

547. Multiplication by a Complex Number. We come now to the general problem of the multiplication of one complex number by another. This includes all other cases as particular cases.

Let $r(\cos\phi + i\sin\phi)$ and $r'(\cos\phi' + i\sin\phi')$ be two complex numbers. By actual multiplication their product is

$$rr'[\cos\phi\cos\phi' - \sin\phi\sin\phi' + i(\sin\phi\cos\phi' + \cos\phi\sin\phi')].$$

By Trigonometry, this may be written

$$rr'[\cos(\phi + \phi') + i\sin(\phi + \phi')].$$

Therefore, the *modulus* of the *product* of two complex numbers is the *product* of their moduli; and the *amplitude* of the product is the *sum* of their amplitudes.

Consequently, the effect of multiplying one complex number by another is *to multiply the modulus of the first by the modulus of the second; and to turn the representative vector of the first through the amplitude of the second.*

548. Division by a Complex Number. The quotient

$$\frac{r(\cos\phi + i\sin\phi)}{r'(\cos\phi' + i\sin\phi')}$$

becomes, multiplying both terms by $\cos\phi' - i\sin\phi'$,

$$\frac{r}{r'}[\cos(\phi - \phi') + i\sin(\phi - \phi')].$$

Consequently, the *modulus* of the quotient of two complex numbers is obtained by *dividing* the modulus of the dividend by that of the divisor; and the *amplitude* of the quotient, by *subtracting* the amplitude of the divisor from that of the dividend.

Jake / f